African Institute for Mathematical Sciences

Problem Solving in Physics 02

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Textbooks

Chabay & Sherwood: Matter & Interactions Vol2
Young & Freedman: University Physics
Halliday & Resnick: Fund. of Physics
Lorrain & Corson: Electromagnetism
Purcell: Electricity and Magnetism

Griffiths: Intro. to Electrodynamics (3rd ed)

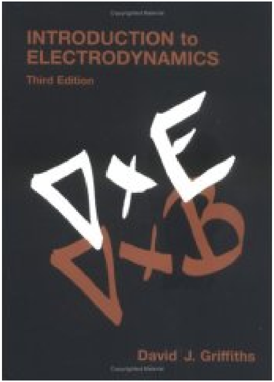
Pollack and Stump: Electromagnetism
Feynman: Lectures in Physics Vol. 2
Nayfeh & Brussel: Electricity and Magnetism
Jackson: Classical Electrodynamics

Vector Calculus: Textbooks

Stewart: Calculus
Swokowski E: Calculus with Analytical Geometry
Griffiths D: Introduction to Electrodynamics
Spiegel M: Vector Analysis (Schaum)
Shercliff J: Vector Fields
Kreyszig E: Advanced Engineering Mathematics
Marsden J & Tromba A: Vector Calculus
Schey: Div, Grad, Curl and All That
Boas M: Math Methods in Physical Sciences
Hague B: Intro to Vector Analysis
Dawber P: Vectors and Vector Operators
Hsu H: Vector Analysis

Chapters in Griffiths 3rd ed

Vector analysis
Electrostatics
Special techniques for potentials
Electric fields in matter
Magnetostatics
Magnetic fields in matter
Electrodynamics
Conservation Laws
Electromagnetic Waves
Potentials and Fields
Radiation
Electrodynamics and Relativity



Maxwell's Equations (integral form)

$$\int_S \mathbf{E} \cdot d\mathbf{a} = \frac{q}{\epsilon_0}$$

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = -\frac{\partial \Phi_B}{\partial t}$$

$$\int_S \mathbf{B} \cdot d\mathbf{a} = 0$$

$$\oint_C \mathbf{B} \cdot d\mathbf{l} = \mu_0 \epsilon_0 \frac{\partial \Phi_E}{\partial t} + \mu_0 i$$

Maxwell's Equations

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{B} = \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} + \mu_0 \mathbf{J}$$

Physics of electrostatics

Coulomb's law and the principle of superposition constitute the physical input for electrostatics – the rest, except for some special properties of matter, is mathematical elaboration of these fundamental rules.

Griffiths: Introduction to Electrodynamics, 3rd ed. p 59

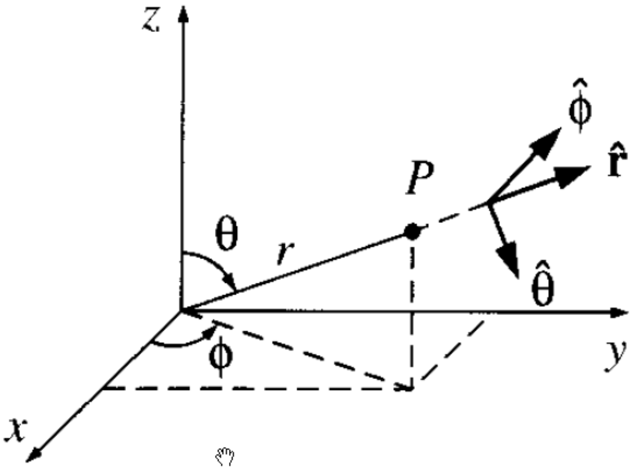
The Electron

In this lecture I wish to give an account of some investigations which have led to the conclusion that the carriers of negative electricity are bodies, which I have called corpuscles, having a mass very much smaller than that of the atom of any known element, and are of the same character from whatever source the negative electricity may be derived.

J.J. Thomson

Nobel lecture, 1906

Spherical polar coordinates



Coulomb's Law

Force between two point charges is proportional to each charge, and inversely proportional to square of distance apart

Force is a vector. Direction? Common sense.

$$\mathbf{F} = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r^2}\hat{\mathbf{r}}$$

Superposition: If many charges, we add them as vectors

$$\mathbf{F_1} = \mathbf{F_{12}} + \mathbf{F_{13}} + \mathbf{F_{14}} + \ldots$$

Note no self-interaction! Hmmm!
Often symmetry helps.

Units. Coulombs, metres; newtons

Permittivity of free space

$$\epsilon_0 = 8.85 \times 10^{-12}\,\mathrm{C}^2\,\mathrm{N}^{-1}\,\mathrm{m}^{-2}$$

Electric field

$\mathbf{E}$ is a vector field. At point P, $\mathbf{r}$ from origin, a *small* test charge q_0 feels a force $\mathbf{F}$

$$\mathbf{E}(\mathbf{r}) = \frac{\mathbf{F}}{q_0}$$

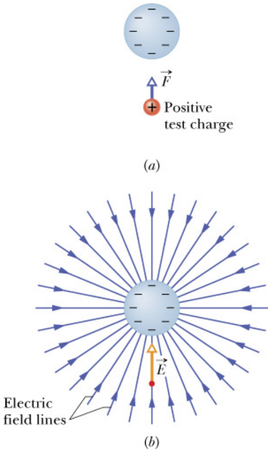
$\mathbf{E}(\mathbf{r})$ is a vector field. Describe by a vector field map or by field lines.

E field of a point charge

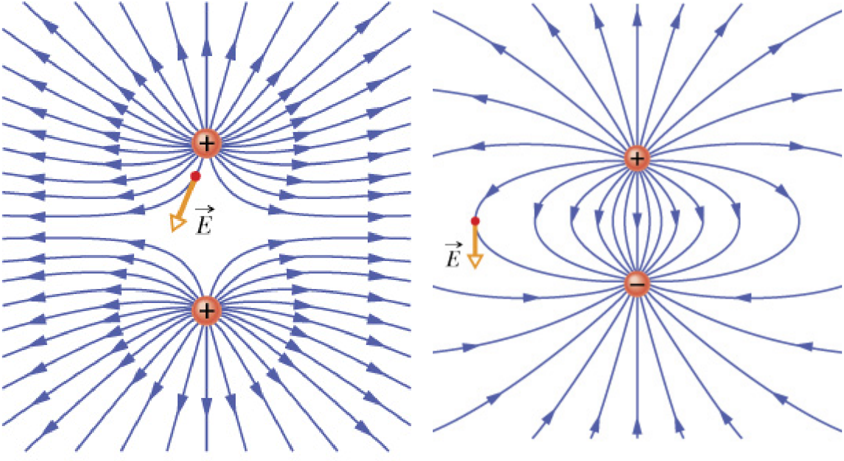
Point charge q at origin.
Spherical symmetry. $\mathbf{E}(\mathbf{r}) = E(r)\hat{\mathbf{r}}$

$$\mathbf{E}(r) = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{\mathbf{r}}$$

Field lines are radial (outwards for $q > 0$).
Density of field lines diminishes like r^{-2} (inverse square).
Beautifully simple!



E field of two point charges

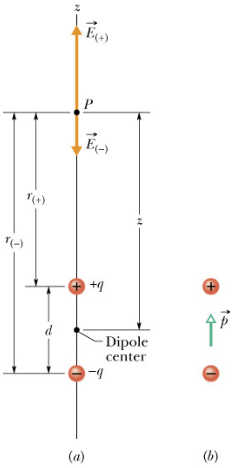


$\mathbf{E}$ field of dipole

$+q$ at $z_1 = +d/2$, $-q$ at $z_2 = -d/2$.
On dipole axis, $z \gg d$

$$\mathbf{E}(z) = +\frac{\hat{\mathbf{z}}}{4\pi\epsilon_0}\frac{(+q)}{(z-d/2)^2} + \frac{\hat{\mathbf{z}}}{4\pi\epsilon_0}\frac{(-q)}{(z+d/2)^2} = \ldots = \frac{\hat{\mathbf{z}}}{4\pi\epsilon_0}\frac{(+2p)}{z^3}$$

where dipole moment $p = qd$

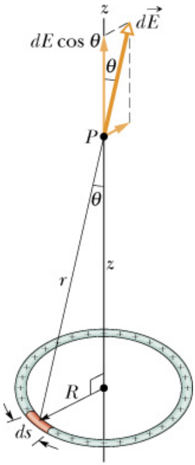


$\mathbf{E}$ field of charged ring

Ring, charge q, radius R. Linear charge density $\lambda = q/(2\pi R)$. On axis

$$\mathbf{E}(z) = \hat{\mathbf{z}} \int dE \cos\theta$$
$$= \frac{\hat{\mathbf{z}}\, z\lambda}{4\pi\epsilon_0(z^2+R^2)^{3/2}} \int_0^{2\pi R} ds$$
$$= \frac{z\lambda(2\pi R)}{4\pi\epsilon_0(z^2+R^2)^{3/2}}\hat{\mathbf{z}}$$

Checks out for $z = 0$, $z \gg R$



$\mathbf{E}$ field from charged semicircular ring

Charge Q, uniformly spread on semicircle of unit radius in (x, y)-plane centred on origin. x-axis is symmetry axis. Point $(-1, 0)$ lies on semicircle. What is $\mathbf{E}$ at origin $(0, 0)$?

Note that $\mathbf{E} = E_x\hat{\mathbf{x}}$ by symmetry, *ie* $E_y = 0$.

Poor: crude approximation: lump all Q at point at middle of arc at $(-1, 0)$. Then $E_x = Q/(4\pi\epsilon_0 r^2)$.

Better: think like a physicist and approximate by considering three equal pieces of arc, each of charge $Q/3$, and lump them at points $(-\cos\pi/3, \sin\pi/3)$, $(-1, 0)$ and $(-\cos\pi/3, -\sin\pi/3)$.

$$
\begin{aligned}
E_x =& E_{1,x} + E_{2,x} + E_{3,x} \\
=& ((Q/3)/(4\pi\epsilon_0)r^2))(\cos(\pi/3) + 1 + \cos(-\pi/3)) \\
=& (Q/(4\pi\epsilon_0 r^2))(2/3)
\end{aligned}
$$

Best: do it by integration.
Consider element subtending $d\theta$. $dQ = Q\,d\theta/\pi$ and

$$\begin{aligned} E_x =& ((Q/\pi)/(4\pi\epsilon_0)r^2) \int_{-\pi/2}^{\pi/2} \cos\theta\, d\theta \\ =& (Q/\pi)/(4\pi\epsilon_0 r^2))[\sin\theta]_{-\pi/2}^{\pi/2} \\ =& (Q/(4\pi\epsilon_0 r^2))(2/\pi) \end{aligned}$$

Physics intuition supports the mathematics.

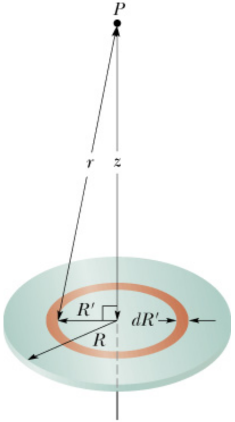
$\mathbf{E}$ field of charged disk

Disk, charge q, radius R. Surf. charge density $\sigma = q/\pi R^2$. Integrate over rings. On axis

$$\mathbf{E}(z) = \hat{\mathbf{z}} \int dE$$

$$= \hat{\mathbf{z}} \frac{\sigma z}{4\pi\epsilon_0} \int_0^R (z^2 + r^2)^{-3/2} (2\pi r)\, dr$$

$$= \hat{\mathbf{z}} \frac{\sigma}{2\epsilon_0} \left(1 - \frac{z}{\sqrt{z^2 + R^2}}\right)$$



For infinite sheet, let $R \to \infty$ $\mathbf{E} = \hat{\mathbf{z}}\, \sigma/(2\epsilon_0)$

The shell theorems

Theorem 1 From outside, a uniform spherical shell of charge behaves as if all charge is at centre

Proof: Invent integral calculus!
OR
use symmetry and flux concept

The shell theorems...

Theorem 2 A uniform spherical shell of charge exerts no force on a charged particle inside the shell.

Proof: Use area of base of cone idea

Really due to geometry of 3-d space, and inverse square law!

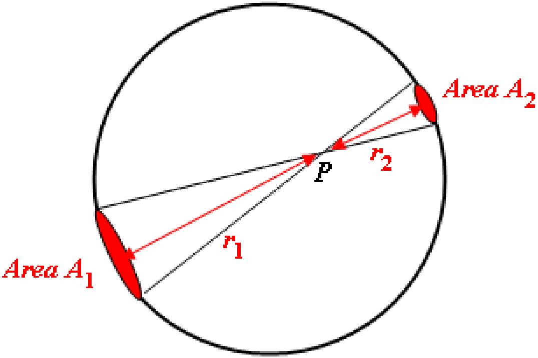
Nature is maximally simple.

Solid angle

2-d world: Line length s, distance R from origin.
Angle subtended: $\theta = s/R$ if $s \ll R$.
Full angle 2π radians

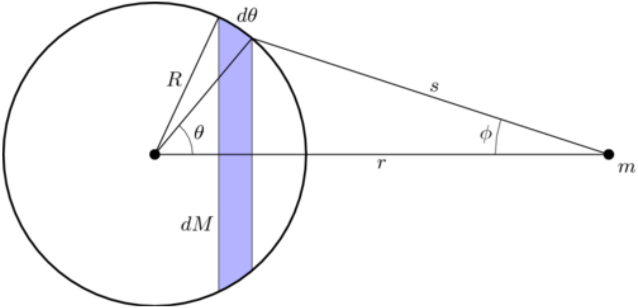
3-d world: Square area s^2 distance R from origin.
Solid angle subtended: $\Omega = s^2/R^2$ if $s^2 \ll R^2$.
Full solid angle 4π steradians

Inverse square law inside spherical shell



Shell theorem

Treat spherically symmetric body as infinite number of concentric, infinitesimally thin spherical shells.



$$dE_r = \frac{kdQ}{s^2}\cos\phi$$

The surface charge density of the entire shell is

$$\sigma = \frac{Q}{4\pi R^2}$$

and the area of the band is

$$dA = 2\pi R^2 \sin\theta \, d\theta$$

making the charge of the band

$$dQ = \sigma \, dA = \tfrac{1}{2} Q \sin\theta \, d\theta$$

The electric field can then be written

$$dE_r = \frac{kQ}{2s^2} \cos\phi \sin\theta \, d\theta$$

By the law of cosines

$$\cos\phi = \frac{r^2 + s^2 - R^2}{2rs}$$

$$\cos\theta = \frac{r^2 + R^2 - s^2}{2rR}$$

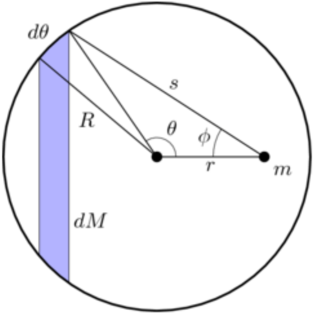
$$\sin\theta \, d\theta = \frac{s}{rR} ds$$

$$dE_r = \frac{kQ}{4r^2R}\frac{r^2 + s^2 - R^2}{s^2}ds$$

To get the total field, we integrate over s as the shaded band sweeps from the point on the sphere closest to q to the farthest (θ goes from 0 to π). Assuming $r > R$:

$$E_r = \frac{kQ}{4r^2R}\int_{r-R}^{r+R}\frac{r^2 + s^2 - R^2}{s^2}ds = \frac{kQ}{r^2}$$

Shell theorem: inside the shell $r < R$



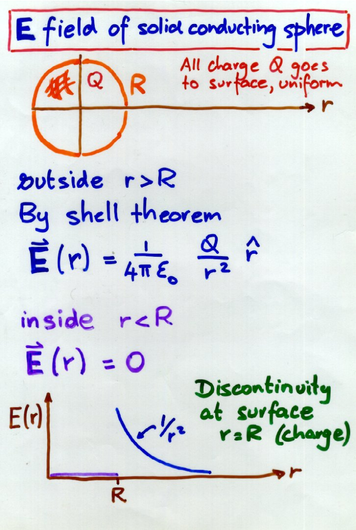
If P is inside shell, ie $r < R$, lower constant of integration is reversed

$$E_r = \frac{kQ}{4r^2R} \int_{R-r}^{R+r} \frac{r^2 + s^2 - R^2}{s^2} ds = 0$$

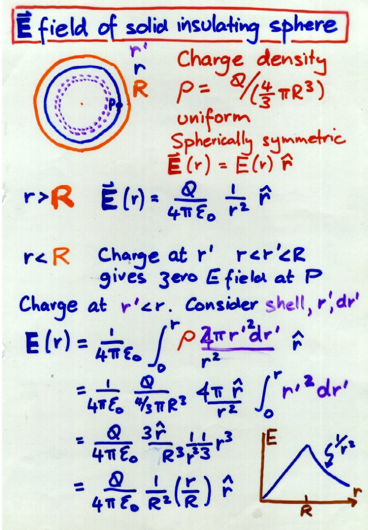
In general, we write:

$$E_r = \begin{cases} kQ/r^2, & r > R \\ 0, & r < R \end{cases}$$

Field due to solid conducting sphere



Field due to solid dielectric sphere



$\vec{E}$ field of solid insulating sphere

Charge density $\rho = Q/(\frac{4}{3}\pi R^3)$

uniform

Spherically symmetric

$\vec{E}(r) = E(r)\,\hat{r}$

$r > R$ $\quad \vec{E}(r) = \frac{Q}{4\pi\varepsilon_0}\frac{1}{r^2}\,\hat{r}$

$r < R$ $\quad$ Charge at r' $\quad r < r' < R$ gives zero E field at P

Charge at $r' < r$. Consider shell, r', dr'

$$E(r) = \frac{1}{4\pi\varepsilon_0}\int_0^r \rho\,\frac{4\pi r'^2 dr'}{r^2}\,\hat{r}$$

$$= \frac{1}{4\pi\varepsilon_0}\,\frac{Q}{\frac{4}{3}\pi R^3}\,\frac{4\pi\,\hat{r}}{r^2}\int_0^r r'^2 dr'$$

$$= \frac{Q}{4\pi\varepsilon_0}\,\frac{3\hat{r}}{R^3}\,\frac{1}{r^2}\,\frac{1}{3}\,r^3$$

$$= \frac{Q}{4\pi\varepsilon_0}\,\frac{1}{R^2}\left(\frac{r}{R}\right)\hat{r}$$

Gauss' Law

Use the idea of electric flux Φ through a closed surface (surface integral of vector field $\mathbf{E}$)

and Gauss' Law

$$\Phi = \frac{Q_{\text{enclosed}}}{\epsilon_0}$$

Gauss' law and symmetry is VERY powerful.

Integral and differential form of Gauss' Law

Integral form of Gauss' Law

$$\Phi = \oint_S \mathbf{E} \cdot d\mathbf{s} = \frac{Q_{\text{enclosed}}}{\epsilon_0} = \frac{1}{\epsilon_0} \int_V \rho \, dv$$

Differential form of Gauss' Law holds at a point

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$$

See later ...

Applications of integral form of Gauss' Law

Gauss' Law always holds. Only useful if there is symmetry

- spherical symmetry (make G surface a concentric sphere)
- cylindrical symmetry (make G surface a long coaxial cylinder)
- plane symmetry (make G surface a pillbox which straddles surface)

See Ex 2.2, Ex 2.3 and Ex 2.4.

E of charged spherical shell

Problem 2.7 Find the electric field a distance z from the center of a spherical surface of radius R (Fig. 2.11), which carries a uniform charge density σ. Treat the case $z < R$ (inside) as well as $z > R$ (outside). Express your answers in terms of the total charge q on the sphere. [*Hint:* Use the law of cosines to write $\mathscr{r}$ in terms of R and θ. Be sure to take the *positive* square root: $\sqrt{R^2 + z^2 - 2Rz} = (R - z)$ if $R > z$, but it's $(z - R)$ if $R < z$.]

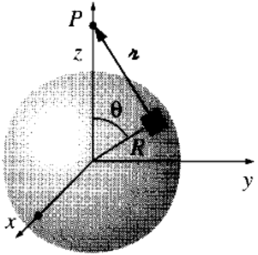
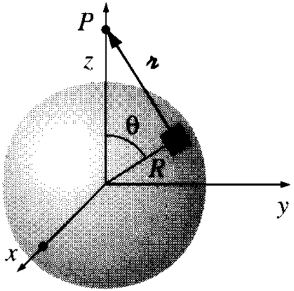


Figure 2.11

$\mathbf{E}$ of charged sphere from $\nabla \cdot \mathbf{E} = \rho/\epsilon_0$



Solid uniformly charged sphere, radius R, at origin
$\rho(r) = \rho$ for $r < R$ and $\rho(r) = 0$ for $r > R$
By symmetry $\mathbf{E} = E_r(r)\,\hat{\mathbf{r}}$

Inside sphere $r < R$

$$\begin{aligned}
\nabla \cdot \mathbf{E} &= \rho/\epsilon_0 \\
\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 E_r) &= \rho/\epsilon_0 \\
\frac{\partial}{\partial r}(r^2 E_r) &= (\rho/\epsilon_0) r^2 \\
r^2 E_r &= (\rho/\epsilon_0)(r^3/3) + C_1 \\
E_r &= (\rho/3\epsilon_0) r + C_1/r^2 \\
E_r &= (\rho/3\epsilon_0) r
\end{aligned}$$

where $C_1 = 0$ because $\mathbf{E} = \mathbf{0}$ at $r = 0$ by symmetry

In terms of total charge $Q = (4/3)\pi R^3 \rho$

For $r < R$

$$\mathbf{E} = \frac{Q}{4\pi\epsilon_0}\frac{1}{R^2}\frac{r}{R}\hat{\mathbf{r}}$$

cont. ...

Outside sphere $r > R$

$$\begin{aligned}
\nabla \cdot \mathbf{E} &= 0 \\
\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 E_r) &= 0 \\
\frac{\partial}{\partial r}(r^2 E_r) &= 0 \\
r^2 E_r &= C_2 \\
E_r &= C_2/r^2 \\
E_r &= (\rho R^3/3\epsilon_0)\, r^{-2}
\end{aligned}$$

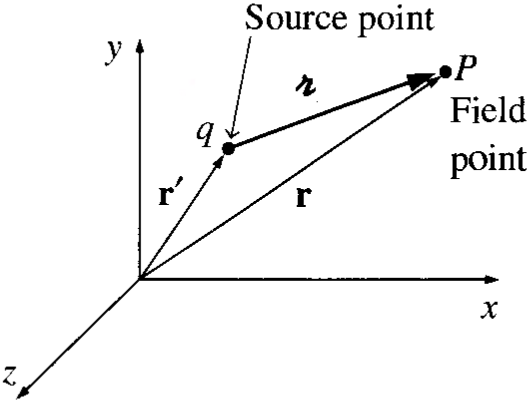
where $C_2 = (\rho/3\epsilon_0)R^3$ since $\mathbf{E}$ is continuous at $r = R$

In terms of total charge $Q = (4/3)\pi R^3 \rho$

For $r > R$

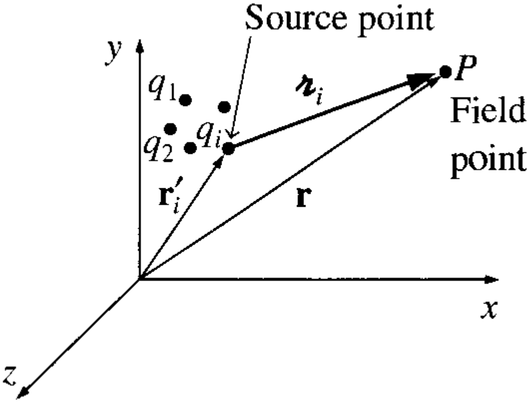
$$\mathbf{E} = \frac{Q}{4\pi\epsilon_0}\frac{1}{r^2}\hat{\mathbf{r}}$$

Field point and source point and $\vec{\mathfrak{r}}$



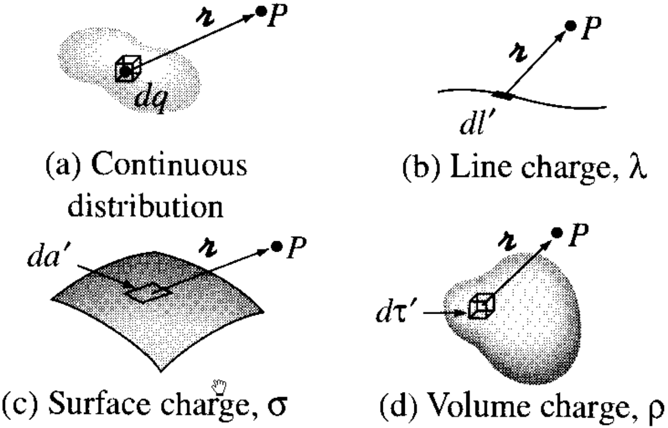
$$\vec{\mathfrak{r}} = \mathbf{r} - \mathbf{r}'$$

Field point and source points and $\vec{\mathfrak{r}}_i$



$$\vec{\mathfrak{r}}_i = \mathbf{r} - \mathbf{r}'_i$$

Integrating over a charge distribution

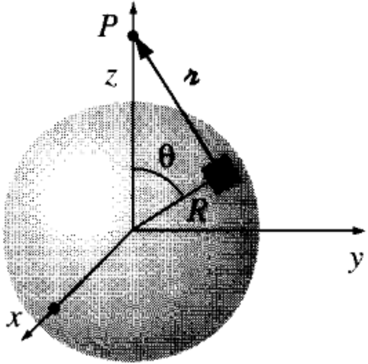


More to come ? . . .

More to come ?? . . . Oh yes.

Inverse square field from spherical shell

Problem 2.7 Find the electric field a distance z from the center of a spherical surface of radius R (Fig. 2.11), which carries a uniform charge density σ. Treat the case $z < R$ (inside) as well as $z > R$ (outside). Express your answers in terms of the total charge q on the sphere. [*Hint:* Use the law of cosines to write $\mathscr{r}$ in terms of R and θ. Be sure to take the *positive* square root: $\sqrt{R^2 + z^2 - 2Rz} = (R - z)$ if $R > z$, but it's $(z - R)$ if $R < z$.]



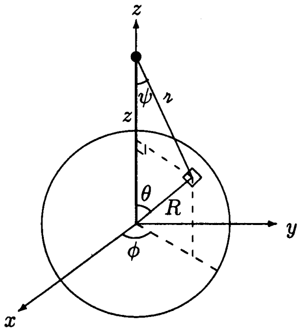
Inverse square field from spherical shell

Problem 2.7

$\mathbf{E}$ is clearly in the z direction. From the diagram,

$dq = \sigma\, da = \sigma R^2 \sin\theta\, d\theta\, d\phi$,
$\imath^2 = R^2 + z^2 - 2Rz\cos\theta$,
$\cos\psi = \frac{z - R\cos\theta}{\imath}$.

So



$$E_z = \frac{1}{4\pi\epsilon_0}\int \frac{\sigma R^2 \sin\theta\, d\theta\, d\phi (z - R\cos\theta)}{(R^2+z^2-2Rz\cos\theta)^{3/2}}. \qquad \int d\phi = 2\pi.$$

$$= \frac{1}{4\pi\epsilon_0}(2\pi R^2\sigma)\int_0^\pi \frac{(z-R\cos\theta)\sin\theta}{(R^2+z^2-2Rz\cos\theta)^{3/2}}\,d\theta. \quad \text{Let } u=\cos\theta;\ du = -\sin\theta\, d\theta; \left\{ \begin{array}{l} \theta = 0 \Rightarrow u = +1 \\ \theta = \pi \Rightarrow u = -1 \end{array} \right\}.$$

$$= \frac{1}{4\pi\epsilon_0}(2\pi R^2\sigma)\int_{-1}^{1} \frac{z-Ru}{(R^2+z^2-2Rzu)^{3/2}}du. \quad \text{Integral can be done by partial fractions—or look it up.}$$

$$= \frac{1}{4\pi\epsilon_0}(2\pi R^2\sigma)\left[\frac{1}{z^2}\frac{zu-R}{\sqrt{R^2+z^2-2Rzu}}\right]_{-1}^{1} = \frac{1}{4\pi\epsilon_0}\frac{2\pi R^2\sigma}{z^2}\left\{\frac{(z-R)}{|z-R|} - \frac{(-z-R)}{|z+R|}\right\}.$$

For $z > R$ (outside the sphere), $E_z = \frac{1}{4\pi\epsilon_0}\frac{4\pi R^2\sigma}{z^2} = \frac{1}{4\pi\epsilon_0}\frac{q}{z^2}$, so $\boxed{\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{z^2}\,\hat{\mathbf{z}}.}$

For $z < R$ (inside), $E_z = 0$, so $\boxed{\mathbf{E} = 0.}$

The shape for maximum gravity

Given a point P in space, and given a piece of malleable material of constant density, how should you shape and place the material in order to create the largest possible gravitational field $\mathbf{H}$ at P?

The shape for maximum gravity

Given a point P in space, and given a piece of malleable material of constant density, how should you shape and place the material in order to create the largest possible gravitational field $\mathbf{H}$ at P?

SOLUTION: Not yet ...

Try

Lagrange multipliers

Program: move bits of surface around till H increases

Think like a physicist! What does maximum mean? ...

The shape for maximum gravity

Given a point P in space, and given a piece of malleable material of constant density, how should you shape and place the material in order to create the largest possible gravitational field $\mathbf{H}$ at P?

SOLUTION: Using Lagrange multipliers and variational calculus

(a) we assume a generic contour curve,

b) we calculate the volume V of the corresponding body,

c) calculate the gravity H at the barypole P, and

d) find the equation for $r(\theta)$ which makes $H = 0$ stationary under the constraint of constant V.

We work out each point:

a) Let $r(\theta)$ be continuous in $[0, \pi/2]$ and possess a derivative anywhere in $(0, \pi/2)$.

b) In the polar coordinates (ρ, θ, ϕ) an infinitesimal volume element is $\mathrm{d}V = \rho^2 \, \sin\theta \, d\rho \mathrm{d}\theta \, \mathrm{d}\phi$

The volume V is

$$\begin{aligned} V &= \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi/2} \int_{\rho=0}^{r(\theta)} \rho^2 \, \sin\theta \, \mathrm{d}r \, \mathrm{d}\theta \, \mathrm{d}\phi \\ &= \ldots \\ &= \frac{2\pi}{3} \int_0^{\pi/2} r^3(\theta) \, \sin\theta \, \mathrm{d}\theta \end{aligned}$$

c) By symmetry H at point P (barypole) is simplified by the symmetry which tells us that H is parallel to the axis. We therefore need to sum up just the parallel components due to all the masses present in the volume elements $\mathrm{d}V$.

$$H = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi/2}\int_{\rho=0}^{r(\theta)} G[\eta(\rho^2\ \sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi)]\frac{\cos\theta}{\rho^2} \quad (1)$$

$$=(2\pi)G\eta\int_0^{\pi/2} r(\theta)\cos\theta\ \sin\theta\,\mathrm{d}\theta \quad (2)$$

Now variational calculus . . . Let $r(\theta) \to r(\theta) + \delta(\theta)$, then $V \to V + \delta V$ and $H \to H + \delta H$

$$\delta V = 2\pi\int_0^{\pi/2}\delta(\theta)r^2(\theta)\ \sin\theta\,\mathrm{d}\theta,\ \delta H = 2\pi G\eta\int_0^{\pi/2}\delta(\theta)r^2(\theta)\ \sin\theta\,\mathrm{d}\theta$$

We want variations that leave the volume constant, $\delta V = 0$ and H to be extreme, $\delta H = 0$.
so $\delta V + \lambda \delta H = 0$ for constant λ

$$\delta V + \lambda \delta H = 2\pi \int_0^{\pi/2} \delta(\theta)[r^2(\theta) + \lambda G\eta \cos\theta] \sin\theta \, \mathrm{d}\theta = 0$$

Thus

$$\lambda = -\frac{1}{G\eta} \frac{\cos\theta}{r^2(\theta)}$$

So λ is constant only if $r(\theta) = a\sqrt{\cos\theta}$

The shape for maximum gravity

Given a point P in space, and given a piece of malleable material of constant density, how should you shape and place the material in order to create the largest possible gravitational field $\mathbf{H}$ at P?

SOLUTION: Think like a physicist.
Assume that the material has been shaped and positioned so that the field at P is maximum. Let this field point in the z-direction. Realize that all the small elements of mass dm on the surface of the material must give equal contributions to the z-component of the field at P. If this were not the case, then we could simply move a tiny piece of the material from one point on the surface to another, thereby increasing the field at P, in contradiction to our assumption that the field at P is maximum.

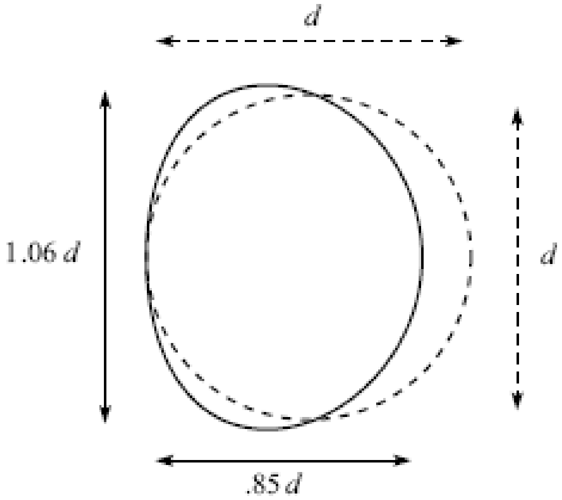
Label the points on the surface by their distance r from P, and by the angle θ that the line of this distance subtends with the z-axis. Then a small mass dm on the surface provides an z-component of the gravitational field equal to

$$F_z = \frac{G\,dm}{r^2}\cos\theta$$

Since this contribution cannot depend on the location of the mass dm on the surface, we must have $r^2 \propto \cos\theta$. The surface may therefore be described by the equation,

$$r^2 = a^2\cos\theta$$

where the constant a^2 depends on the volume of the material. The desired shape clearly exhibits cylindrical symmetry around the z-axis, so let us consider a cross section in the z–y plane.


d
1.06 d
d
.85 d

In terms of z and y, with $z^2 + y^2 = r^2$ and $\cos\theta = z/r$, eq. (2) becomes

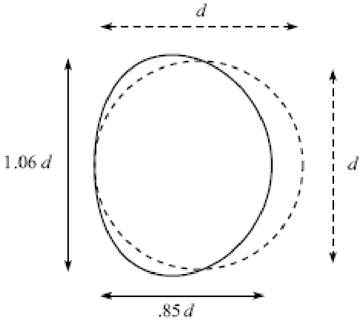
$$r^3 = a^2 z \quad \Rightarrow \quad r^2 = a^{4/3} z^{2/3} \quad \Rightarrow \quad y^2 = a^{4/3} z^{2/3} - z^2$$

To get a sense of what this surface looks like, note that $dy/dz = 0$ at both $z = 0$ and $z = a$ (the point on the surface furthest from P). The surface is smooth with no cusps. The volume is

$$V = \int_0^a \pi y^2 \, dz = \int_0^a \pi \left(a^{4/3} z^{2/3} - z^2 \right) \mathrm{d}z = \frac{4\pi}{15} a^3$$

Since a sphere of volume V has diameter $d = (6V/\pi)^{1/3}$, we see that a sphere with the same volume would have a diameter of $(8/5)^{1/3} a \approx 1.17a$. Hence our shape is squashed by a factor of $(5/8)^{1/3} \approx 0.85$ along the z-direction, compared to a sphere of the same volume.

We may also calculate the maximum height in the y-direction. You can show that is occurs at $z = 3^{-3/4}a \approx 0.44a$ and has a value of $2(4/27)^{1/3}a \approx 1.24a$. Hence in the y direction our shape is stretched by a factor of $2(4/27)^{1/4}(5/8)^{1/3} \approx 1.24/1.17 \approx 1.06$ compared to a sphere of the same volume. Cross sections of our shape and a sphere with the same volume are shown below.



What is $\mathbf{H}$? Calculate . . .

End AIMS PSP 02

Thank you.