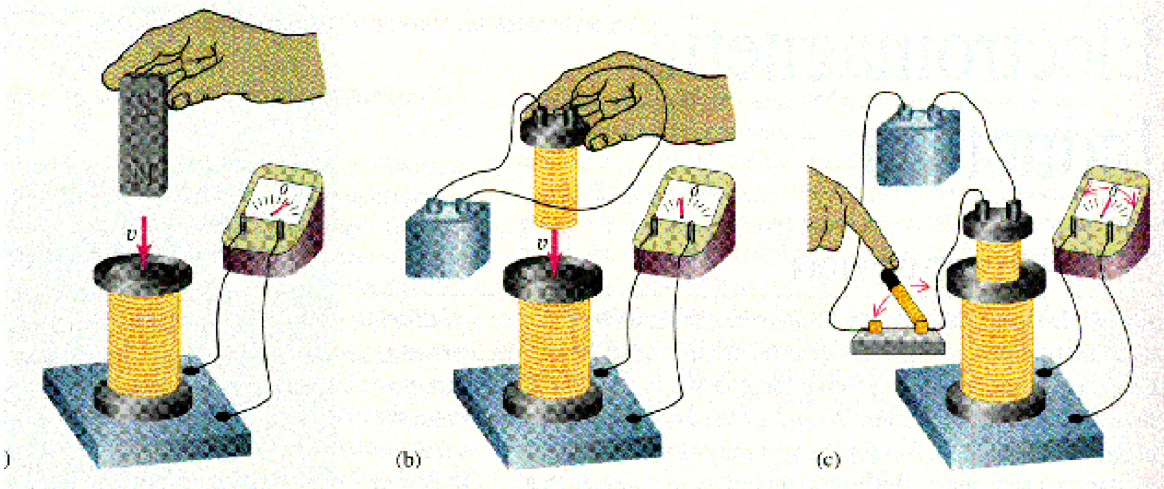
Electromagnetic Induction

Most electric motors can be operated in reverse - as dynamos to generate electricity. Just as we can put a current carrying conductor in a magnetic field to produce motion, we can move a conductor in a magnetic field to produce an **induced current**.



Alternatively,

- we can move a magnet near a coil, or
- replace the magnet with a current carrying coil, or even
- change the current in the current carrying coil.

If the magnet and coils are stationary, or the current in the coil is constant, the induced current is zero.

In each case the induced current results from an **induced emf** which is caused by a **changing magnetic flux**.

Faraday's Law

Faraday's Law relates the emf ξ, induced in a current loop, to the rate of change of the magnetic flux $\Phi_B = \int \mathbf{B} \cdot d\mathbf{A}$.

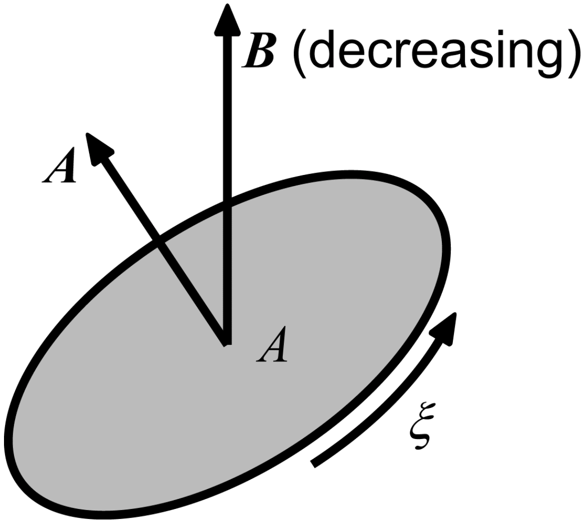
$$\boxed{\xi = -\frac{d\Phi_B}{dt}} \qquad (31\text{-}6)$$

For a coil of N identical turns, we have N sources of emf in series. The total induced emf is therefore

$$\boxed{\xi = -N\frac{d\Phi_B}{dt}} \qquad (31\text{-}7)$$

Lenz's Law

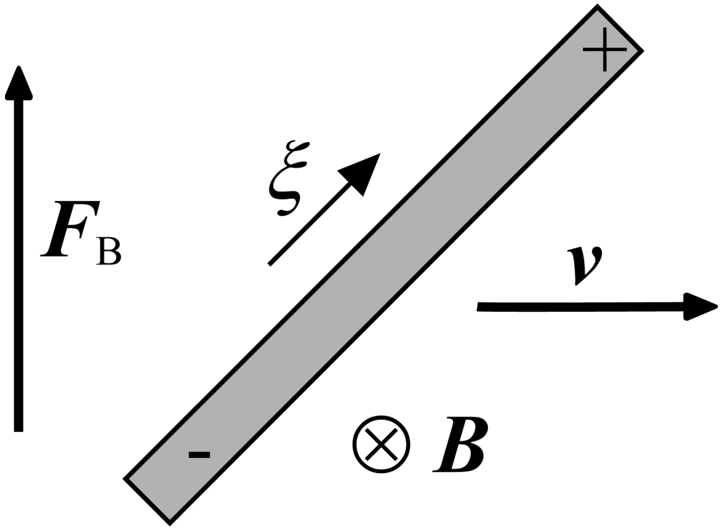
Conventionally ξ is proportional to the rate of decrease of Φ_{B}. Its direction (clockwise around the direction of $\boldsymbol{A}$) is such that the magnetic field due to the induced current reinforces the external field, as given by Lenz's Law:



The direction of any magnetic induction effect is such as to oppose the cause of the effect.

Motional emf

The induced emf results from the magnetic force acting on the charges in a conductor due to a *relative* motion between the charges and the field. We do **not** need a closed loop in which to induce the emf.



Consider an element of a conductor $d\mathbf{l}$ inclined at an angle to its direction of motion in a magnetic field.

The magnetic force on each charge carrier is $\boldsymbol{F}_{\mathrm{B}} = q\,\boldsymbol{v} \times \boldsymbol{B}$,

which will be balanced by an electrostatic force $\boldsymbol{F}_{\mathrm{E}} = q\boldsymbol{E}$,

from the electric field caused by the redistribution of charges in the wire.

Setting the two forces equal and eliminating the charge gives

$$\boxed{\boldsymbol{v} \times \boldsymbol{B} = \boldsymbol{E}.}$$

The potential difference between the ends of the line element is then

$$\boxed{\xi = \boldsymbol{E} \cdot \boldsymbol{dl} = (\boldsymbol{v} \times \boldsymbol{B}) \cdot \boldsymbol{dl}\ .}$$

If we were to connect this element to a stationary circuit, it would act as a source of emf, driving a current from its more positive side through the circuit, and back through the negative terminal. The direction of this **motional emf** is therefore in the direction of $d\boldsymbol{l}$ from the negative terminal to the positive.

For a complete wire, the motional emf is then

$$\boxed{\xi = \int (\boldsymbol{v} \times \boldsymbol{B}) \cdot d\boldsymbol{l}}$$

For a straight wire of length L, moving perpendicularly to both its length and a uniform magnetic field, the motional emf is simply $\xi = vBL$.

Induced Electric Fields

A changing magnetic flux induces an emf, which in a conducting loop results in a current, even in the parts of the circuit where the flux is constant.

If we break the loop, or remove the conductor completely, we no longer have a current, but there is still an induced emf!

Any potential difference can be expressed in terms of an electric field. It is this **induced electric field** which drives the current in the conductor.

For any path in the field, the potential difference is given by the integral of the field.

$$\boxed{\xi = \oint \boldsymbol{E} \cdot d\boldsymbol{l}}$$

We can reformulate Faraday's Law as

$$\boxed{\oint \mathbf{E} \cdot d\mathbf{l} = -\frac{d\Phi_B}{dt}} \qquad (31\text{-}22)$$

The induced electric field is a **non-electrostatic field**, and differs from an electrostatic field in two important respects.

- There are no free charges associated with it, and
- it is non-conservative.

As a consequence of this, induced electric field lines form closed loops (usually circles). If we move a charged particle around a closed path back to its original position, the change in potential is $2\pi rE$ (for a circular path) every time we go around.

The potential is **not uniquely defined at any point**.

Note: a similar problem arises with magnetic fields, which are also not conservative.

Eddy Currents

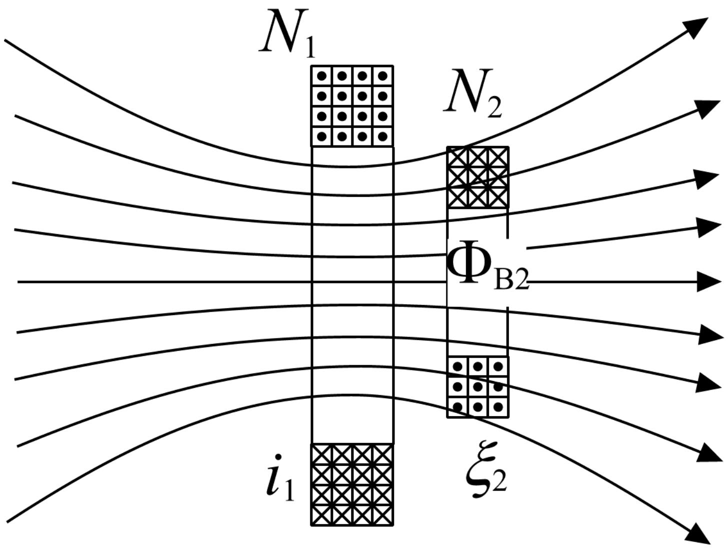
If we replace the single current loop by a large conductor, the induced electric field will cause circular currents throughout the material. These currents resemble eddies in turbulent flow, and are known as **eddy currents**.

The rotational direction of the currents eddy currents will be the same as for a single loop, acting according to Lenz’s law to resist the change in magnetic flux.

If the change in flux results from motion, the interaction between the applied and induced magnetic fields will be to reduce the relative velocity of the magnetic field and conductor. This has applications in induction motors and braking systems.

Inductance

If we consider two coils, labelled 1 and 2, placed in proximity to one another. A time varying current i_1 in the first coil will produce a time varying magnetic field. Part of the flux of this field, Φ_{B2}, will pass through the second coil. As this flux changes it will induce an emf ξ_2 in the second coil.



Mutual Inductance

If the second coil has N_2 turns, the induced emf will be

$$\boxed{\xi_2 = -N_2 \frac{d\Phi_{B2}}{dt}}$$

Assuming that the positions and geometries of the coils do not change the flux Φ_{B2} will be proportional to the current i_1 in the first coil. Introducing the **mutual inductance** M_{21}

$$\boxed{N_2\Phi_{B2} = M_{21}i_1\ .}$$

Combining these two equations, we obtain

$$\boxed{\xi_2 = -M_{21}\frac{di_1}{dt}} \tag{31-63}$$

If, instead, an emf ξ_1 was induced in the first coil by a current i_2 in the second coil, this would be

$$\boxed{\xi_1 = -M_{12}\frac{di_2}{dt}} \tag{31-64}$$

The mutual inductance depends only on the geometry of both coils, as well as any magnetic materials between them, and on the number of turns in both coils. For a few special geometries, *e.g.* concentric solenoids or two coils on an iron core, it is easy to show that $M_{12} = M_{21}$.

Generalising, we can omit the subscript

(31-66) $$\boxed{\xi_2 = -M\frac{di_1}{dt}}$$ and $$\boxed{\xi_1 = -M\frac{di_2}{dt}}$$ (31-67)

where $$\boxed{M = \frac{N_1\Phi_{B2}}{i_1} = \frac{N_2\Phi_{B1}}{i_2}}$$

The unit of inductance is the henry (H)

1 H = 1 Tm^2 A^{-1} = 1 Wb A^{-1} = 1 V s A^{-1} = 1 Ω s.

Self-Inductance and Inductors

Even in a single coil, a changing magnetic flux due to a varying current will induce an emf. This **self-induced emf** will, by Lenz’s law oppose the change in current in the coil.

As for mutual inductance, we can define the **self-inductance**

$$\boxed{L = \frac{N\Phi_B}{i}} \qquad (31\text{-}30)$$

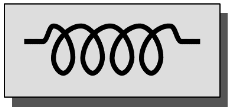
Rearranging this, and taking the time derivative

$$\boxed{N\frac{d\Phi_B}{dt} = L\frac{di}{dt}}$$

Substituting into Faraday’s Law, yields the self-induced emf

$$\boxed{\xi = -N\frac{d\Phi_B}{dt} = -L\frac{di}{dt}} \qquad (31\text{-}37)$$

A device which has a defined inductance is known as an **inductor**, and usually has the symbol



Magnetic Field Energy

Just as a capacitor stores energy in its internal electric field, an inductor stores energy in its magnetic field.

If the inductor has zero resistance, the potential difference across the inductor is $V = -\xi$.

At any time, the power supplied to the device is

$$\boxed{P = Vi = -\xi i = Li\frac{di}{dt}}$$

and the increase in energy supplied during a time interval dt

$$\boxed{dU_B = Pdt = Li\,di\,.} \qquad (31\text{-}50)$$

The total energy required to increase the current from zero to i is

$$\boxed{U_B = L\int_0^i i\,di = \frac{1}{2}Li^2} \qquad (31\text{-}51)$$

In the same way as for the electric field, we can represent this magnetic potential energy as a stored energy density per unit volume for the field.

If, for example the inductor is a long solenoid, of length d, cross-sectional area A, and n turns per unit length, the flux is

$$\boxed{\Sigma_{\mathrm{B}} = BA = \mu_0 nIA\ .}$$

The inductance of the solenoid is then

$$\boxed{L = \frac{N\Phi_B}{i} = \mu_0 n^2 lA}$$

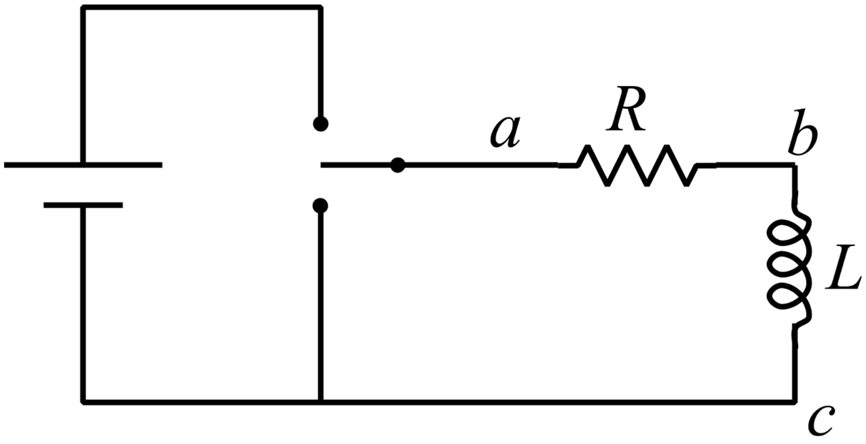
from which the stored energy is

$$\boxed{U_B = \frac{1}{2}LI^2 = \frac{\mu_0 n^2 I^2}{2} \times lA}$$

Recognising that lA is the volume of the magnetic field, and substituting for the field strength, $B = \mu_0 nI$, the energy density per unit volume is

$$\boxed{u_B = \frac{\mu_0 n^2 I^2}{2} = \frac{B^2}{2\mu_0}} \qquad (31\text{-}56)$$

The R-L Circuit



At some time t after the switch has connected the resistor and inductor to the battery, the current in the circuit is i.

The instantaneous potential differences across the two components are

$$\boxed{v_{ab} = iR \qquad \text{and} \qquad v_{bc} = L\,di/dt\ .}$$

Applying Kirchhoff’s loop rule

$$\boxed{\xi - iR - L\frac{di}{dt} = 0} \qquad (31\text{-}40)$$

Initially (at $t=0$)

$$\boxed{\left(\frac{di}{dt}\right)_{t=0} = \frac{\xi}{L}}$$

which is independent of the resistance.

After a long time, the current reaches a constant value

$$\boxed{I = \frac{\xi}{R}}$$

which is independent of the inductance.

At intermediate times, we have (as for the charge on a capacitor)

$$\boxed{\frac{di}{1 - \xi/R} = -\frac{R}{L}dt}$$

The solution of this is $\ln\left(\dfrac{i-\xi/R}{-\xi/R}\right)=-\dfrac{R}{L}t$

or $$\boxed{i=\frac{\xi}{R}\left(1-\exp\left(-\frac{R}{L}t\right)\right)} \qquad (31\text{-}43)$$

Again, we have a characteristic **time constant** for the circuit

$$\boxed{\tau = L/R\ .} \qquad (31\text{-}44)$$

If we now switch the circuit so the resistor and inductor are no longer connected to the source of emf, the current decays with the same characteristic time.

$$\boxed{i=\frac{\xi}{R}\exp\left(-\frac{R}{L}t\right)=i_0e^{-t/\tau}} \qquad (31\text{-}47)$$