PHY2014F Labs

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http://www.phy.uct.ac.za/courses/phylab2/

Books

Buffler et al. *Introduction to measurement in the Physics Laboratory.* (ex PHY1004W)

GUM ANSI/ISO *Guide to the Expression of Uncertainty in Measurement*.

Bevington *Data reduction and error analysis for the physical sciences.*

Sivia *Data analysis — a Bayesian tutorial.*

Measurement

In statistical terms we can distinguish two types of measurement:

- parameter estimation
- hypothesis testing.

What they have in common is that the results rely on the measurements made by the experimenter.

Measurement

The thing being measured (the measurand) presumably has some true value μ.

The task of experiment is to determine an estimate of this, say $\bar{X}$.

The central problem of experimental physics is to determine μ (the "law of physics") from the measured values X_i via some estimate $\bar{X}$.

Uncertainties

Measurements have associated uncertainties.

The uncertainty of a measurement

is a parameter, associated with the result of a measurement, that characterises the dispersion of the values that could reasonably be attributed to the quantity being measured.

Sources of uncertainty (GUM)

1. incomplete definition of the measurement;
2. imperfect realisation of the definition of the measurand;
3. nonrepresentative sampling — the sample measured may not represent the defined measurand;
4. inadequate knowledge of the effects of environmental conditions on the measurements or imperfect measurement of environmental effects;
5. personal bias in reading analogue instruments;

Sources of uncertainty (GUM)

6. finite instrument resolution or discrimination threshold;
7. inexact values of measurement standards and reference materials;
8. inexact values of constants and other parameters obtained from external sources and used in the data reduction algorithm; approximations and assumptions incorporated in the measurement method and procedure;
9. variations in repeated observations of the measurand under apparently identical conditions.

Uncertainties

We distinguish two types of error leading to uncertainty.

- **Random errors.**

  Measurements cannot be made arbitrarily precise. The limitations of the measurement apparatus cause measurements to have some inherent uncertainty.

  A series of measurements of the same quantity will display some spread of values.

  This type of error can be analysed statistically.

Uncertainties

- **Systematic errors.**
  These arise from uncertainties or mistakes in calibration of apparatus or in experimental design.

  For instance, a voltmeter might read 10% low, and this would affect all results in a well-defined, non-random way.

Treatment of uncertainty

- Type A uncertainties: those that can be evaluated by statistical means.

  — usually means lots of measurements can be averaged.

- Type B uncertainties: those evaluated by some means other than statistical.

  — Some prior knowledge about the measurement technique is needed.

Measurements

There are three steps that need to be considered in making a measurement:

- calculation of a result
- evaluation of the standard uncertainties in the result
- determining the reliablility of the measurement by association of an *expanded uncertainty*.

  This defines an interval in which we can expect a large fraction p of the possible results to lie, where p is a coverage probability or level of confidence.

Mean

N measurements of some quantity x, say $x_1, x_2, \ldots x_N$.

We define the (arithmetic) mean of the measurements as

$$\bar{x} = \frac{1}{N} \sum_{1}^{N} x_i$$

We take the mean as the best estimate of the “actual” value. An alternative notation is

$$\langle x \rangle \equiv \bar{x}$$

Variance

A measure of the “average spread” in measurements.
Define the variance as

$$\sigma^2 = \frac{1}{N}\sum(x_i - \mu)^2$$

Since we don’t know μ a more useful quantity is

$$s^2 = \frac{1}{N-1}\sum(x_i - \bar{x})^2$$

This is an ‘unbiased estimator of the variance’.

Standard deviation

The quantity $s =$ is known as the standard deviation.

This gives a measure of the spread in values of the x_i about the mean.

It is not the uncertainty in our knowledge of the mean $\bar{x}$.

Standard deviation of the mean

This is given by $s' = s/\sqrt{N}$.

- The standard deviation tells us something about the uncertainties in individual measurements.
- The standard deviation of the mean tells us about the uncertainty in the average of a number of measurements.

It also tells us that more measurements implies smaller errors.

Confidence interval

Results are usually expressed as values together with a confidence interval.

This is an indication of how reliable we feel the result is.

These intervals are usually expressed as 68%, 95% or 99% confidence intervals.
(“one–,two– or three–sigma”).

Normal distribution

For a continuous quantity, the distribution in random uncertainties is often given by the normal (also called Gaussian) distribution.

$$P(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp -(x-\mu)^2/2\sigma^2$$

Normal distribution

The value μ is the mean of the distribution and also gives the position of the maximum.

The variance is σ^2.

The area below the curve and between the limits $(\mu - \sigma)$ and $(\mu + \sigma)$ is 0.68 and that between $(\mu - 2\sigma)$ and $(\mu + 2\sigma)$ is just over 0.95.

Confidence intervals

For a small number N of observations, the estimates of s and s' are themselves uncertain, and we must increase the size of the interval to compensate.
This can be combined with the numerical factor to replace the confidence interval with $(\bar{x} - ts', \bar{x} + ts')$ where t depends on N and the degree of confidence.

e.g., 95% CI, $N = 3$, $t = 4.30$. $N = 10$, $t = 2.26$

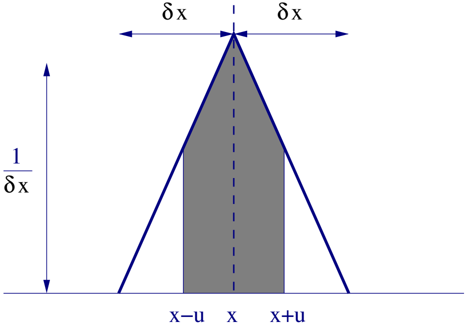
Type B uncertainties

We might argue that δx is the maximum value of the uncertainty, but we're pretty sure the actual uncertainty is smaller.

This might be appropriate if we interpolate between the graduations on the metre stick. ("It's a little closer to that mark, so . . . 0.9517 m")

Type B uncertainties

We might then assume we have a triangular distribution, and the standard uncertainty is $\delta x/\sqrt{6}$.



Combining uncertainties

Experimental quantities will nearly always depend on a number of measured quantities each with their own uncertainties.

In order to determine the uncertainty in the result these uncertainties must be combined in an appropriate way.

Combining uncertainties — example

Suppose that $x = y - z$ and the variances of y and z are s_y^2 and s_z^2.
Then the variance in x is

$$
\begin{aligned}
s_x^2 &= \left\langle (x - \bar{x})^2 \right\rangle \\
&= \left\langle [(y - z) - (\bar{y} - \bar{z})]^2 \right\rangle \\
&= \left\langle (y - \bar{y})^2 + (z - \bar{z})^2 - 2(y - \bar{y})(z - \bar{z}) \right\rangle \\
&= \left\langle (y - \bar{y})^2 \right\rangle + \left\langle (z - \bar{z})^2 \right\rangle - 2 \langle (y - \bar{y})(z - \bar{z}) \rangle \\
&= s_y^2 + s_z^2 - 2\mathrm{cov}(y, z)
\end{aligned}
$$

Combining uncertainties — sums

Suppose we take N (independent) measurements of x. Each x_i has a variance of s_x^2. We can determine the variance in the sum of all of these measurements,

$$\sum_1^N x_i = N\bar{x}$$

by using the above method in the case of addition,

$$s_{N\bar{x}}^2 = \sum_1^N s_x^2 = N s_x^2$$

Combining uncertainties — error in mean

We can also determine this by having a number of measurements of of $N\bar{x}$ which is of course an experimental quantity.

$$\begin{aligned} s^2_{N\bar{x}} &= \left\langle (N\bar{x} - N\bar{\bar{x}})^2 \right\rangle \\ &= N^2 \left\langle (\bar{x} - \bar{\bar{x}})^2 \right\rangle \\ &= N^2 s^2_{\bar{x}} \\ &= N s^2_x \end{aligned}$$

i.e. $N s^2_{\bar{x}} = s^2_x$

Combining uncertainties — general

Consider a function $f(x)$, assume uncertainties are small.
An uncertainty in x leads to an uncertainty in f

$$\Delta f = \frac{\mathrm{d}f}{\mathrm{d}x}\Delta x$$

where the derivative is evaluated at $\bar{x}$. For a set of such measurements we obviously have

$$s_f^2 = \left(\frac{\mathrm{d}f}{\mathrm{d}x}\right)^2 s_x^2$$

Combining uncertainties — several variables

This can now be extended by addition to several variables, since for $f(x, y, \ldots)$ we have

$$\Delta f = \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y + \ldots$$

and

$$s_f^2 = \left(\frac{\partial f}{\partial x}\right)^2 s_x^2 + \left(\frac{\partial f}{\partial y}\right)^2 s_y^2 + \ldots$$

where the covariance terms can be included if necessary.

Combining uncertainties

	standard deviation	fractional s. d.
$x \pm y$	$(s_x^2 + s_y^2)^{1/2}$	
kx	ks_x	$s_x/\bar{x}$
xy		$[(s_x/\bar{x})^2 + (s_y/\bar{y})^2]^{1/2}$
x^p		$ps_x/\bar{x}$
$f(x, y, \dots)$	$\left[\left(\frac{\partial f}{\partial x}\right)^2 s_x^2 + \left(\frac{\partial f}{\partial y}\right)^2 s_y^2 + \dots\right]^{1/2}$	

Method of least squares

Function $y_{th}(i, a_1, a_2, \ldots)$ where a_j are unknown parameters, and i represents a set of variables describing the point at which the measurement is made. A statistic is defined:

$$\chi^2 = \sum_{i=1}^{N} \frac{(y_i - y_{th}(i, a_1, a_2, \ldots))^2}{e_i^2}.$$

Then the method of least squares asserts that this 'chisquared' value will be a minimum for the 'best fit' parameters.

Fitting a line

Consider the simplest non-trivial case, the straight line, $y_{th} = ax + b$. (In statistics, this is known as linear regression).

Then

$$\chi^2 = \sum_{i=1}^{N} \frac{(y_i - ax_i - b)^2}{e_i^2}.$$

Here, the e_i are the errors associated with the measured y_i.

Fitting a line

At the minimum in chisquare, i.e. for the best-fit parameters,

$$\frac{\partial \chi^2}{\partial a} = 0 = -2 \sum \frac{x_i y_i}{e_i^2} + 2a \sum \frac{x_i^2}{e_i^2} + 2b \sum \frac{x_i}{e_i^2}$$

and

$$\frac{\partial \chi^2}{\partial b} = 0 = -2 \sum \frac{y_i}{e_i^2} + 2a \sum \frac{x_i}{e_i^2} + 2b \sum \frac{1}{e_i^2}$$

Fitting a line

Introducing

$$S = \sum \frac{1}{e_i^2}, \qquad S_x = \sum \frac{x_i}{e_i^2}, \qquad S_y = \sum \frac{y_i}{e_i^2},$$

$$S_{xx} = \sum \frac{x_i^2}{e_i^2}, \qquad S_{xy} = \sum \frac{x_i y_i}{e_i^2}$$

Fitting a line

The solution is found to be:

$$a = \frac{S S_{xy} - S_x S_y}{\Delta}$$

$$b = \frac{S_y S_{xx} - S_x S_{xy}}{\Delta}$$

where $\Delta = S S_{xx} - S_x^2$.

Fitting a line

Using the usual formula for propogation of errors, the variances of a and b can be found:

$$s_a^2 = \frac{S_{xx}}{\Delta}$$

$$s_b^2 = \frac{S}{\Delta}$$

$$\mathrm{Cov}(a, b) = \frac{-S_x}{\Delta}$$