CHAPTER 3

Neutron Distributions in Energy

3.1 Introduction

In Chapter 1 we briefly introduced the concept of neutron multiplication, defining it as

\[ k = \frac{\text{number of neutrons in } i\text{th + 1 generation}}{\text{number of neutrons in } i\text{th generation}}, \]  

(3.1)

where neutrons in a particular generation are considered to be born in fission, undergo a number of scattering collisions, and die in absorption collisions. Understanding what determines the magnitude of the multiplication is central to the study of neutron chain reactions. This chapter examines the determinants of multiplication with primary emphasis placed on the kinetic energy of the neutrons, for as we saw in Chapter 2, the basic data—the neutron cross sections—are strongly energy dependent. These energy dependencies define the two broad classes of reactors: thermal and fast. We first discuss the properties of nuclear fuel and of materials that moderate the neutron spectrum. With this background we proceed to provide a more detailed description of the energy distributions of neutrons in nuclear reactors, and then discuss the averaging of neutron cross sections over energy. We conclude by defining the neutron multiplication in terms of energy-averaged cross sections.

This chapter’s discussions maintain two simplifications in order to focus the analysis on the energy variable. First, we assume that all neutrons are produced instantaneously at the time of fission, postponing discussion of the effects created by the small fraction of neutrons whose emission following fission is delayed until the detailed treatment of reactor kinetics in Chapter 5. Second, we defer analysis of the spatial distributions of neutrons in nuclear reactors to later chapters. For now we take the finite size of a reactor into account.
simply by noting that for many systems the multiplication can by approximated by

\[ k = k_\infty P_{NL}, \]  

(3.2)

where \( P_{NL} \) is the neutron nonleakage probability and \( k_\infty \) is the multiplication that would exist if a reactor’s dimensions were infinitely large. Chapters 6 and 7 treat neutron leakage and other spatial effects in some detail. Our focus here is on neutron energy, and how the energy dependence of the cross sections dominates the determination of \( k_\infty \).

### 3.2 Nuclear Fuel Properties

Much of the physics of nuclear reactors is determined by the energy dependence of the cross sections of fissile and fertile materials over the range of incident neutron energies between the fission spectrum and the Maxwell-Boltzmann distribution of thermal neutrons, thus over the range between roughly 10 MeV and 0.001 eV. Recall that fissile nuclides have significant fission cross sections over this entire range as indicated by Figs. 2.9 and 2.10 for those of uranium-235 and plutonium-239. In contrast, fission of a fertile material is possible only for incident neutrons above some threshold; Fig. 2.8 indicates that the threshold for uranium-238 is at approximately 1.0 MeV. Not all of the neutrons absorbed by a fissile nucleus will cause fission. Some fraction of them will be captured, and that fraction is also energy dependent. As a result, the number of fission neutrons produced per neutron absorbed plays a central role in determining a reactor’s neutron economy:

\[ \eta(E) = \frac{\nu \Sigma_f(E)}{\Sigma_a(E)} = \frac{\text{fission neutrons produced}}{\text{neutrons absorbed}}, \]  

(3.3)

where \( \nu \) is the number of neutrons produced per fission, and

\[ \Sigma_a(E) = \Sigma_\gamma(E) + \Sigma_f(E). \]  

(3.4)

To sustain a chain reaction, the average value of \( \eta(E) \) must be substantially more than one, for in a power reactor neutrons will be lost to absorption in structural, coolant, and other materials, and some will leak out of the system.

To examine the behavior of \( \eta(E) \) we first consider a single fissile isotope. We may then cancel the number densities from numerator and denominator of Eq. [3.3] to obtain
\[\eta(E) = \frac{\nu \sigma_f(E)}{\sigma_a(E)}. \tag{3.5}\]

From the plots of \(\eta(E)\) shown in Fig. 3.1 for uranium-235 and plutonium-239 we see that concentrating neutrons at either high or low energies and avoiding the range between roughly 1.0 eV and 0.1 MeV where the curves dip to their lowest values most easily achieves a chain reaction. Except for naval propulsion systems designed for the military, however, fuels consisting predominately of fissile material are not employed in power reactors. Enrichment and fabrication costs would render them uneconomical. More importantly, the fuel would

![Graphs of \(\eta(E)\) for fissile isotopes](image.png)

**FIGURE 3.1** \(\eta(E)\) for fissile isotopes (courtesy of W. S. Yang, Argonne National Laboratory). (a) Uranium-235, (b) Plutonium-239.
constitute weapons grade uranium or plutonium that would compound the problems of nuclear proliferation. Reactor fuels consist primarily of uranium-238 with a smaller fraction of fissile material, referred to as its enrichment. Depending on the design, civilian reactor fuels normally consist of uranium with enrichments ranging from the 0.7% of natural uranium up to approximately 20% fissile material.

To determine $\eta(E)$ for a reactor fuel, we first define the enrichment $\varepsilon$ as the atom ratio of fissile to fissionable (i.e., fertile plus fissile) nuclei:

$$\varepsilon = \frac{N_{f_i}}{N_{f_e} + N_{f_i}},$$  

where $f_i$ and $f_e$ denote fissile and fertile. Equation (3.3) then reduces to

$$\eta(E) = \frac{\varepsilon \sigma_f^f(E) + (1 - \varepsilon) \nu \sigma_f^f(E)}{\varepsilon \sigma_a^f(E) + (1 - \varepsilon) \sigma_a^f(E)}.$$  

Figure 3.2 provides plots of $\eta(E)$ for natural (0.7%) and 20% enriched uranium. These curves illustrate the dramatic effect that the capture cross section of uranium-238 has in deepening the valley in $\eta(E)$ through the intermediate energy range. Conversely, above its threshold value 1.0 MeV, the increasing fission cross section of uranium-238 aids strongly in increasing the value of $\eta(E)$. The curves emphasize why power reactors are classified as fast or thermal.

![Figure 3.2](image-url)
according to the energy spectra over which neutrons are concentrated, and why no intermediate spectrum reactors have been built.

Reactor designs must concentrate neutrons either in the fast or the thermal energy range and thus avoid the sharp valley that $\eta(E)$ exhibits over intermediate energies. As Chapter 2 emphasizes, scattering collisions cause neutrons to lose energy until they approach equilibrium in the thermal neutron range. Thus for fast reactor cores, designers eliminate materials other than the fuel as much as possible. They avoid low atomic weight materials in particular, for elastic scattering in such materials quickly reduces neutron energies to levels to where resonance capture in uranium-238 predominates. Even if all other materials could be eliminated, however, a fast reactor fueled with natural uranium is not possible, for the large inelastic scattering cross section of the 99.3% uranium-238 would then cause the fission neutrons to fall too quickly into the intermediate energy range. Consequently, fast reactors require fuels enriched to more than 10%.

For thermal reactors the situation is reversed. The reactor must contain a substantial quantity of low atomic weight material, referred to as a moderator. Its purpose is to slow down neutrons past the valley in $\eta(E)$ with relatively few collisions to the thermal energies where the fuel’s ratio of neutron production to absorption again exceeds one by a substantial margin. With optimized ratios of moderator to fuel, thermal reactors can be designed with much lower enrichments than fast reactors; with some moderators—most notably graphite or heavy water—thermal reactors may be fueled with natural uranium. To understand why this is so we must examine the properties of moderators more closely.

### 3.3 Neutron Moderators

In thermal reactors moderator materials are required to reduce the neutron energies from the fission to the thermal range with as few collisions as possible, thus circumventing resonance capture of neutrons in uranium-238. To be an effective moderator a material must have a low atomic weight. Only then is $\xi$—the slowing down decrement defined by Eq. (2.54)—large enough to slow neutrons down to thermal energies with relatively few collisions. A good moderator, however, must possess additional properties. Its macroscopic scattering cross section must be sufficiently large. Otherwise, even though a neutron colliding with it would lose substantial energy, in the competition with other materials, too few moderator collisions would take place to have a significant impact on the neutron spectrum. Thus a second important parameter in determining a material’s value as a moderator is the slowing down power, defined as $\xi \Sigma_s$, where $\Sigma_s$ is the macroscopic scattering cross section.
where $\Sigma_s = N\sigma_s$ is the macroscopic scattering cross section. Note that the number density $N$ must not be too small. Thus gases are eliminated. Helium, for example, has sufficiently large values of $\xi$ and $\sigma_s$ to be a good moderator but its number density is too small to have a significant impact on the energy distribution of neutrons in a reactor. Conversely, for the same reason gases such as helium may be considered as coolants for fast reactors since they do not degrade the neutron spectrum appreciably.

Table 3.1 lists values of the slowing down decrement and power for the three most common moderators. The table also includes the slowing down ratio: the ratio of the material’s slowing down power to its thermal absorption cross section. If the thermal absorption cross section $\Sigma_a(E_{\text{thermal}})$ is large, a material cannot be used as a moderator; even though it may be effective in slowing down neutrons to thermal energy, it will then absorb too many of those same neutrons before they can make collisions with the fuel and cause fission. Note that heavy water has by far the largest slowing down ratio, followed by graphite and then by ordinary water. Power reactors fueled by natural uranium can be built using D$_2$O as the moderator. Because graphite has poorer moderating properties, the design of natural uranium fueled power reactors moderated by graphite is a more difficult undertaking. Reactors using a light water moderator and fueled with natural uranium are not possible; some enrichment of the uranium is required to compensate for the larger thermal absorption cross section of the H$_2$O.

Large thermal absorption cross sections eliminate other materials as possible moderators. For example, boron-10 has reasonable values of the slowing down decrement and power. Its thermal absorption cross section, however, is nearly 4000 barns. As a result boron cannot be used as a moderator but is, in fact, one of the more common neutron “poisons,” which are used to control or shut down the chain reactions.

The foregoing discussion focuses on elastic scattering, since inelastic scattering tends to be of much less importance in determining the energy distribution of neutrons in thermal reactors. The lighter weight materials either have no inelastic scattering cross sections that are large enough to be significant, or they tend to be inefficient moderators.
section, or if they do, they are zero below a threshold that is quite high in energy. Fertile and fissile materials do have inelastic scattering above thresholds in the keV to MeV range. In thermal reactors inelastic collisions only modestly augment the slowing down by elastic scattering. The situation is quite different in fast reactors where the absence of moderator material causes inelastic scattering to become more important. Inelastic scattering of the fuel along with elastic scattering with the coolant and structural materials are the primary causes for unwanted energy spectrum degradation.

3.4 Neutron Energy Spectra

To recapitulate, the distribution of neutrons in energy is determined largely by the competition between scattering and absorption reactions. For neutrons with energies significantly above the thermal range, a scattering collision results in degradation of the neutron energy, whereas neutrons in thermal equilibrium have near equal probabilities of gaining or losing energy when interacting with the thermal motions of the nuclei that constitute the surrounding medium. In a medium for which the average energy loss per collision and the ratio of scattering to absorption cross section are both large, the neutron distribution in energy will be close to thermal equilibrium and is then referred to as a soft or thermal spectrum. Conversely in a system with small ratios of neutron degradation to absorption, neutrons are absorbed before significant slowing down takes place. The neutron distribution then lies closer to the fission spectrum and is said to be hard or fast.

The neutron distribution may be expressed in terms of the density distribution

$$\tilde{n}''(E)dE = \begin{cases} \text{number of neutrons/cm}^3 \\ \text{with energies between } E \text{ and } E + dE, \end{cases} \quad (3.8)$$

which means that

$$n'' = \int_0^\infty \tilde{n}''(E)dE = \text{total number of neutrons/cm}^3. \quad (3.9)$$

The more frequently used quantity, however, is the neutron flux distribution defined by

$$\varphi(E) = \nu(E)\tilde{n}''(E), \quad (3.10)$$

where $\nu(E)$ is the neutron speed corresponding to kinetic energy $E$. 
The flux, often called the scalar flux, has the following physical interpretation: \( \varphi(E)dE \) is the total distance traveled during one second by all neutrons with energies between \( E \) and \( dE \) located in 1 cm\(^3\). Likewise, we may interpret the macroscopic cross section as

\[
\Sigma_x(E) = \begin{cases} 
\text{Probability/cm of flight of a neutron} \\
\text{with energy } E \text{ undergoing a reaction of type } x.
\end{cases}
\] (3.11)

Thus multiplying a cross section by the flux, we have

\[
\Sigma_x(E)\varphi(E)dE = \begin{cases} 
\text{Probable number of collisions of type } x/s/cm^3 \\
\text{for neutrons with energies between } E \text{ and } dE.
\end{cases}
\] (3.12)

Finally, we integrate over all energy to obtain

\[
\int_0^\infty \Sigma_x(E)\varphi(E)dE = \begin{cases} 
\text{Probable number of collisions} \\
\text{of type } x/s/cm^3 \text{ of all neutrons.}
\end{cases}
\] (3.13)

This integral is referred to as a reaction rate, or if \( x = s, a, f \) as the scattering, absorption, or fission rate.

A more quantitative understanding of neutron energy distributions results from writing down a balance equation in terms of the neutron flux. Since \( \Sigma(E)\varphi(E) \) is the collision rate—or number of neutrons of energy \( E \) colliding/s/cm\(^3\)—each such collision removes a neutron from energy \( E \) either by absorption or by scattering to a different energy. We may thus regard it as a loss term that must be balanced by a gain of neutrons arriving at energy \( E \). Such gains may come from fission and from scattering. The number coming from fission will be \( \chi(E) \), given by Eq. (2.31). We next recall that the probability that a neutron that last scattered at energies between \( E_0 \) and \( E_0 + dE_0 \) will be scattered to an energy \( E \) as \( p(E' \rightarrow E)dE' \). Since the number of neutrons scattered from energy \( E' \) is \( \Sigma_s(E')\varphi(E') \), the scattering contribution comes from integrating \( p(E' \rightarrow E)\Sigma_s(E'\varphi(E')dE' \) over \( E' \). The balance equation is thus

\[
\Sigma_s(E)\varphi(E) = \int p(E' \rightarrow E)\Sigma_s(E')\varphi(E')dE' + \chi(E)s''(E).
\] (3.14)

The specific form of \( p(E' \rightarrow E) \) for elastic scattering by a single nuclide is given by Eq. (2.47), whereas Eq. (2.53) defines the composite probability for situations where the cross sections are sums over more than one nuclide. For brevity we write the foregoing equation as
where as in Eq. (2.50) we take $\Sigma_s(E' \rightarrow E) = p(E' \rightarrow E)\Sigma_s(E')$. The balance equation is normalized by the fission term, which indicates a rate of $s^{''''}_f$ fission neutrons produced/s/cm$^3$.

Using Eq. (3.15) to examine idealized situations over three different energy ranges provides some insight into the nature of neutron spectra, particularly of thermal reactors. First, we consider fast neutrons, whose energies are sufficient that $\chi(E)$ is significant. Normally the lower limit to this range is about 0.1 MeV. We then examine intermediate energy neutrons, which have energies below the range where fission neutrons are produced but sufficiently high that up-scatter—that is, energy gained in a collision as a result of the thermal motion of the scattering nuclide—can be ignored. The lower cutoff for intermediate neutrons is conventionally taken as 1.0 eV. The intermediate energy range is often referred to as the resonance or slowing down region of the energy spectra because of the importance of these two phenomena. Third, we discuss slow or thermal neutrons defined as those with energies less than 1.0 eV; at the lower energies thermal motions of the surrounding nuclei play a predominant role in determining the form of the spectrum. In each of the three energy ranges general restrictions apply to Eq. (3.15). In the thermal and intermediate ranges no fission neutrons are born and thus $\chi(E) = 0$. In the intermediate and fast ranges there is no up-scatter, and therefore $\Sigma_s(E' \rightarrow E) = 0$ for $E' < E$.

**Fast Neutrons**

Over the energy range where fission neutrons are born both terms on the right of Eq. (3.15) contribute; near the top of that range the fission spectrum $\chi(E)$ dominates, since on average even one scattering collision will remove a neutron to a lower energy. In that case we may make the rough approximation,

$$\varphi(E) \approx \chi(E)s^{''''}_f/\Sigma_t(E),$$

which only includes the uncollided neutrons: those emitted from fission but yet to make a scattering collision. Even in the absence of moderators or other lower atomic weight materials the spectrum will be substantially degraded as a result of inelastic scattering collisions with uranium or other heavy elements. The presence of even small amounts of lighter weight materials, such as the metals used within a reactor core for structural support, adds to the degradation of the fast spectrum. Neutron moderators, of course greatly accelerate
the slowing down of neutrons out of the fast range. In fast reactors, where lightweight materials are avoided, most of the neutrons are absorbed before scattering collisions slow them down below the low-energy tail of the fission spectrum.

**Neutron Slowing Down**

We next examine the energy range that extends below where $\chi(E)$ is significant but higher than the thermal energy range, where the thermal motions of the nuclei must be taken into account.

**The Slowing Down Density**

A useful concept for treating neutrons in this energy range is the slowing down density, which we define as

$$ q(E) = \begin{cases} \text{number of neutrons slowing down past energy } E/s/cm^3. 
\end{cases} \quad (3.17) $$

At energies greater than where up-scatter occurs, any neutron produced by fission that is not absorbed at a higher energy must slow down past that energy. Thus

$$ q(E) = - \int_E^\infty \Sigma_a(E')\varphi(E')dE' + \int_E^\infty \chi(E')dE' s''', \quad E > 1.0 \text{ eV}. \quad (3.18) $$

In the intermediate range, below where fission neutron production is significant, the normalization of $\chi(E')$ given by Eq. (2.32) simplifies Eq. (3.18) to

$$ q(E) = - \int_E^\infty \Sigma_a(E')\varphi(E')dE' + s''', \quad 1.0 \text{ eV} < E < 0.1 \text{ MeV}. \quad (3.19) $$

Taking the derivative, we have

$$ \frac{d}{dE} q(E) = \Sigma_a(E)\varphi(E). \quad (3.20) $$

Thus $q(E)$ decreases as the neutrons slow down in proportion to the absorption cross section; if there is no absorption over some energy interval then the slowing down density remains constant.

In the intermediate range the primary form of absorption comes from the resonance capture cross sections discussed in Chapter 2. However, between those resonances the absorption cross section is small enough to be ignored. Thus between resonances we see from Eq. (3.20) that the slowing down density is independent of energy.
Moreover, since we are below the energies where fission neutrons are produced, with no absorption, Eq. (3.14) simplifies to

\[ \Sigma_s(E)\varphi(E) = \int p(E' \rightarrow E)\Sigma_s(E')\varphi(E')dE'. \]  
(3.21)

Thus we can obtain a particularly simple relationship between \( \varphi(E) \) and \( q \), the constant slowing down density. We next assume that we are below the threshold for inelastic scattering, and that only a single scattering material—normally a moderator—is present. (We may later modify the expression for combinations of materials.) Equation (2.47) provides the kernel for elastic scattering. Substituting it into Eq. (3.21) yields

\[ \Sigma_s(E)\varphi(E) = \int_{E}^{E/\alpha} \frac{1}{(1 - \alpha)E'}\Sigma_s(E')\varphi(E')dE'. \]  
(3.22)

The solution may be shown to be

\[ \Sigma_s(E)\varphi(E) = C/E \]  
(3.23)

by simply inserting this expression into Eq. (3.22).

The normalization constant \( C \) is proportional to \( q \), the number of neutrons slowed down by scattering past energy \( E \). Examining Fig. 3.3, we observe that the number of neutrons that made their last scatter at \( E' > E \) to energies \( E'' < E \) will fall in the interval \( \alpha E' \leq E'' \leq E \). Moreover, only neutrons with initial energies \( E' \) between \( E \) and \( E/\alpha \) are capable of scattering to energies below \( E \). Hence the number of neutrons slowing down past \( E \) per cm\(^3\) in one second is

\[ q = \int_{E}^{E/\alpha} \left[ \int_{E/\alpha}^{E} \frac{1}{(1 - \alpha)E'}\Sigma_s(E')\varphi(E')dE'' \right]dE'. \]  
(3.24)

Substituting Eq. (3.23) for the flux, and performing the double integration, we obtain

\[ q = \left[ 1 + \frac{\alpha}{1 - \alpha} \ln \alpha \right] C. \]  
(3.25)
Noting that the bracketed term is identical to \( \xi \), the slowing down decrement defined by Eq. (2.56), we may combine Eqs. (3.23) and (3.25) to represent the flux in terms of the slowing down density

\[
\varphi(E) = \frac{q}{\xi \Sigma_s(E) E}.
\]

(3.26)

This expression may be extended to situations where more than one scattering nuclide is present by adding their contributions to Eqs. (3.22) and (3.24). Suppose both fuel and moderator are present. Equation (3.26) still holds, where the scattering cross section becomes the sum over fuel and moderator, and the slowing down decrement is replaced by the weighted average defined by Eq. (2.61):

\[
\bar{\xi} = \frac{\xi f \Sigma_f(E) + \xi m \Sigma_m(E)}{\Sigma_f(E) + \Sigma_m(E)}.
\]

(3.27)

Between resonances the fuel and moderator scattering cross sections are nearly independent of energy. The flux is then proportional to \( 1/E \)—and referred to as a “one-over-\( E \)” flux. Since the moderator is much lighter than the fuel, \( \xi f \ll \xi m \), the fuel contribution to Eq. (3.26) is much less than that of the moderator.

**Energy Self-Shielding**

In the presence of resonance absorber, the flux is no longer proportional to \( 1/E \). However, we may obtain a rough estimate of its energy dependence by making some reasonable approximations. We assume that only fuel and moderator are present, and that only elastic scattering takes place. Equation (3.14) then reduces to

\[
\Sigma_r(E) \varphi(E) = \int_{E/\alpha_f}^{E/\alpha_f} \frac{1}{(1 - \alpha_f)E'} \Sigma_f(E') \varphi(E') dE' + \int_{E/\alpha_m}^{E/\alpha_m} \frac{1}{(1 - \alpha_m)E'} \Sigma_m(E') \varphi(E') dE',
\]

(3.28)

where for the energy range of resonance absorbers we have set \( \chi(E) = 0 \). Recall from Chapter 2 that a resonance is characterized by a width \( \Gamma \). If the resonances are widely spaced, then the bulk of the resonance absorption will take place within about \( \pm \Gamma \) of the resonance energy. Moreover, outside this interval absorption can be ignored and the flux approximated as \( \propto 1/E \).

Scattering into the energy interval where absorption is most pronounced originates over a larger energy interval: between \( E \) and
In the narrow resonance approximation, which is valid for all but a few resonances, we assume both of these intervals to be much larger than the resonance width, as shown schematically in Fig. 3.4. In this case the preponderance of the areas under the integrals in Eq. (3.28) are occupied by the $1/E$ flux between resonances where absorption can be ignored and the scattering cross sections are energy independent. Thus we may insert Eq. (3.26) into the right side of Eq. (3.28) without much loss of accuracy. We evaluate the integrals with energy-independent constant cross section to obtain

$$
\varphi(E) = \frac{q}{\xi \Sigma_t(E) E}, \quad (3.29)
$$

where $q$ is the neutron slowing down density above the resonance. Note that the only difference from Eq. (3.26) is that in the denominator the scattering has been replaced by the total cross section.

The total cross section, of course, includes both resonance absorption and scattering cross sections. Thus it increases greatly, causing the flux to decrease correspondingly, at energies where resonance absorption takes place. Such flux depression—illustrated in Fig. 3.4—is referred to as energy self-shielding. According to Eqs. (3.19) and (3.20), as neutrons slow down through a resonance the slowing down density is reduced by

$$
\int \varphi(E) \Sigma_a(E) dE \approx \int \frac{\Sigma_a(E)}{\Sigma_t(E) E} dE \frac{q}{\xi}. \quad (3.30)
$$

Since self-shielding reduces the flux where the absorption cross section is large, it reduces overall neutron losses to absorption, and thus aids the propagation of the chain reaction. In the following chapter we will see that by lumping the fuel spatial self-shielding of the resonances serves further to reduce the absorption losses of neutrons.
approximations can be applied when the resonance width is wider, but the qualitative effects of energy self-shielding remain the same.

**Thermal Neutrons**

At lower energies, in the thermal neutron range, we again use Eq. (3.15) as our starting point. The fission term on the right vanishes. The source of neutrons in this case comes from those scattering down from higher energies. We may represent this as a scattering source. We divide the integral in Eq. (3.15) according to whether \( E \) is less than or greater than the cutoff energy for the thermal neutron range, typically taken as \( E_0 = 1.0 \text{ eV} \). We then partition the equation as

\[
\Sigma_t(E) \varphi(E) = \int_0^{E_0} \Sigma_s(E' \rightarrow E) \varphi(E') dE' + s(E) q_0, \quad E < E_0, \tag{3.31}
\]

where

\[
s(E) q_0 = \int_{E_0}^{\infty} \Sigma_s(E' \rightarrow E) \varphi(E') dE', \quad E < E_0, \tag{3.32}
\]

is just the source of thermal neutrons that arises from neutrons making a collision at energies \( E' > E_0 \), but having an energy of \( E < E_0 \) after that collision. The source may be shown to be proportional to the slowing down density at \( E_0 \), and if pure scattering and a \( 1/E \) flux is assumed at energies \( E' > E_0 \), a simple expression results for \( s(E) \), the energy distribution of the source neutrons. In the thermal range the scattering distribution is difficult to represent in a straightforward manner, for not only thermal motion, but also binding of the target nuclei to molecules or within a crystal lattice must be factored into the analysis.

We may gain some insight by considering the idealized case of a purely scattering material. Then the solution of Eq. (3.31) would become time dependent, for without absorption in an infinite medium the neutron population would grow continuously with time since each slowed down neutron would go on scattering forever. If after some time the slowing down density were set equal to zero, an equilibrium distribution would be achieved satisfying the equation

\[
\Sigma_s(E) \varphi_M(E) = \int_0^{E_0} \Sigma_s(E' \rightarrow E) \varphi_M(E') dE'. \tag{3.33}
\]

One of the great triumphs of kinetic theory was the proof that for this equation to be satisfied, the principle of detained balance must be obeyed. Detailed balance states that
\[
\Sigma_s(E \rightarrow E') \phi_M(E) = \Sigma_s(E' \rightarrow E) \phi_M(E'),
\]

(3.34)

no matter what scattering law is applicable. Equally important, the principle states that in these circumstances the flux that satisfied this condition is the form found by multiplying the famed Maxwell-Boltzmann distribution, given by Eq. (2.34), by the neutron speed to obtain

\[
\phi_M(E) = \frac{1}{(kT)^2} E \exp(-E/kT)
\]

(3.35)

following normalization to

\[
\int_0^\infty \phi_M(E) dE = 1.
\]

(3.36)

In reality some absorption is always present. Absorption shifts the thermal neutron spectrum upward in energy from the Maxwell-Boltzmann distribution, since complete equilibrium is never reached before neutron absorption takes place. Figure 3.5 illustrates the upward shift, called spectral hardening, which increases with the size of the absorption cross section. Nevertheless, Eq. (3.35) provides a rough approximation to a reactor’s thermal neutron distribution. A somewhat better fit to hardened spectra, such as those in Fig. 3.5, may be obtained by artificially increasing the temperature \( T \) by an amount that is proportional to \( \Sigma_a/\Sigma_s \).

![FIGURE 3.5 Thermal spectra compared to a Maxwell-Boltzmann distribution (adapted from A. F. Henry, Nuclear-Reactor Analysis, 1975, by permission of the MIT Press).](image-url)
Fast and Thermal Reactor Spectra

Figure 3.6 shows typical neutron spectra plotted as $E\varphi(E)$ for a sodium-cooled fast reactor and for a water-cooled thermal reactor. Several features are noteworthy. Fast reactor spectra are concentrated in the keV and MeV range with nearly all of the neutrons absorbed before slowing down to energies less than a keV. Fast reactor cores contain intermediate weight elements, such as sodium coolant and iron used for structural purposes. These intermediate atomic weight elements have large resonances in their elastic scattering cross sections in the keV and MeV energy range. Thus the fast spectra are quite jagged in appearance, resulting from the energy self-shielding phenomenon, illustrated by Eqs. (3.16) and (3.29), in which the flux is inversely proportional to the total cross section.

Thermal reactor spectra have a more modest peak in the MeV range where fission neutrons are born. The spectra over higher energies are somewhat smoother as a result of the prominent role played by the lightweight moderator materials; moderators have no resonances at those energies, and therefore the cross sections in the denominators of Eqs. (3.16) and (3.29) are smoother functions of energy. Moving downward through the keV range, we see that the spectrum is nearly flat. Here there is very little absorption, resulting in a nearly $1/E$ [or constant $E\varphi(E)$] spectrum with the constant

![Figure 3.6 Neutron flux spectra from thermal (pressurized water) and fast (sodium-cooled) reactors (courtesy of W. S. Yang, Argonne National Laboratory).](image-url)
slowing down density as given by Eq. (3.26). The thermal reactor spectra do decrease with decreasing energy going from 100 and 1.0 eV, accentuated by sharp dips in the flux. Although barely visible in the figure, resonance absorption in uranium over this energy range causes the slowing down density to decrease and the self-shielding indicated in Eq. (3.29) to become more pronounced. Below 1.0 eV, the characteristic thermal peak occurs. As a result of thermal neutron absorption in the fuel and moderator, the peak in the thermal spectrum is at an energy somewhat higher than would be indicated by the Maxwell-Boltzmann distribution given by Eq. (3.35). Finally, note that if we had plotted $\varphi(E)$ instead of $E\varphi(E)$ for the thermal reactor, the thermal flux peak would be millions of times larger than the peak of fission energy neutrons.

### 3.5 Energy-Averaged Reaction Rates

As the foregoing sections indicate, the ability to sustain a chain reaction depends a great deal on the distribution of neutrons in energy, which in turn is determined by the composition of nonfissile materials in the core and their effectiveness in slowing down the neutrons from fission toward thermal energies. To determine the overall characteristics of a reactor core, we must average cross sections and other data over the energy spectrum of neutrons. We accomplish this through the use of Eq. (3.13), which is termed the reaction rate for collisions of type $x$ and has units of collisions/s/cm$^3$.

Reaction rates are commonly expressed as products of energy-averaged cross sections and the neutron flux:

$$
\int_0^\infty \Sigma_x(E)\varphi(E)dE = \bar{\Sigma}_x\phi,
$$

where the cross section is

$$
\bar{\Sigma}_x = \int_0^\infty \Sigma_x(E)\varphi(E)dE / \int_0^\infty \varphi(E)dE,
$$

and the flux, integrated over energy, is

$$
\phi = \int_0^\infty \varphi(E)dE.
$$

For a known neutron flux distribution, microscopic cross sections may also be averaged over energy. We simply make the replacement $\Sigma_x = N\sigma_x$ in Eqs. (3.37) and (3.38) to eliminate the atom density and obtain
and

\[
\bar{\sigma}_x = \int_0^\infty \sigma_x(E)\Phi(E) dE \int_0^\infty \Phi(E) dE.
\]  

(3.41)

We may also express the flux as the product of the mean speed and the density of the neutrons:

\[
\Phi = \bar{\nu} \tilde{n}^\prime,
\]

(3.42)

where Eq. (3.9) defines the neutron density \( n^\prime \). To accomplish this insert the flux definition given by Eq. (3.10) into Eq. (3.39):

\[
\Phi = \int_0^\infty \nu(E)\tilde{n}^\prime(E) dE
\]

(3.43)

and note that to be consistent with Eq. (3.42) the mean speed must be defined by

\[
\bar{\nu} = \int_0^\infty \nu(E)\tilde{n}^\prime(E) dE \int_0^\infty \tilde{n}^\prime(E) dE.
\]

(3.44)

Frequently we will drop the bar indicating averaging from the left sides of Eqs. (3.38) and (3.41). Thus hereafter we assume that a cross section \( \Sigma_x \) or \( \sigma_x \) appearing without the \( (E) \) attached has been averaged over energy. If a cross section is independent of energy, we have \( \sigma_x(E) \rightarrow \sigma_x \), and then, of course, we may take it outside the integral in Eqs. (3.38) and (3.41), and we have simply, \( \bar{\sigma}_x = \sigma_x \) and \( \Sigma_x = \Sigma_x \).

More refined treatments of a neutron population often require cross section averaging over some limited range of neutron energies rather than over the entire neutron energy spectrum. The discussions of Section 3.4 indicate that the analysis of neutron spectra fall naturally into thermal, intermediate, and fast energy ranges. Correspondingly we may partition reaction rates as

\[
\int \sigma_x(E)\Phi(E) dE = \int_T \sigma_x(E)\Phi(E) dE + \int_I \sigma_x(E)\Phi(E) dE
\]

\[
+ \int_F \sigma_x(E)\Phi(E) dE,
\]

(3.45)

where hereafter attaching \( T, I, \) and \( F \) signifies integration over the ranges \( 0 \leq E \leq 1.0 \text{ eV}, 1.0 \text{ eV} \leq E \leq 0.1 \text{ MeV}, \) and \( 0.1 \text{ MeV} \leq E \leq \infty \), respectively. Employing Eq. (3.40), we may write this sum in terms of energy-averaged cross sections as
\[ \bar{\sigma}_x \phi = \bar{\sigma}_{xT} \phi_T + \bar{\sigma}_{xI} \phi_I + \bar{\sigma}_{xF} \phi_F. \] (3.46)

Each of the terms on the right results from multiplying and dividing the corresponding integral of Eq. (3.45) by

\[ \phi_\Omega = \int_\Omega \varphi(E) dE, \quad \Omega = T, I, F, \] (3.47)

and defining the energy averaged cross sections as

\[ \bar{\sigma}_{x\Omega} = \int_\Omega \sigma_x(E) \varphi(E) dE \left/ \int_\Omega \varphi(E) dE \right., \quad \Omega = T, I, F. \] (3.48)

More advanced so-called multigroup methods divide the energy spectrum into more than the three intervals shown here, and considerable effort is expended in determining the flux spectra in each group as accurately as possible. For our purposes, however, the division into thermal, intermediate, and fast energy segments is adequate. We perform cross section averaging by selecting appropriate flux approximations for use in Eqs. (3.46) through (3.48). We begin with the fast neutrons and work our way downward in energy.

**Fast Cross Section Averages**

Even though it includes only uncollided neutrons, Eq. (3.16) provides a first approximation to the flux distribution for fast neutrons. The total macroscopic cross section in the denominator, however, includes all of the nuclides present—fuel, coolant, and so on. Thus it is likely to be a strong and complex function of energy, particularly if significant concentrations of iron, sodium, or other elements that have scattering resonances in the MeV range are present. In Fig. 3.6 these effects are apparent in the jagged appearance of the fast flux for both thermal and fast reactors. To preclude the cross sections tabulated for individual elements from being dependent on the other elements present, we must further simplify Eq. (3.16) by taking \( \Sigma_i(E) \) as energy independent. Then normalizing to \( s_i^{eff}/\Sigma_i = 1.0 \), we have \( \varphi(E) \approx \chi(E) \). Since only a very small fraction of fission neutrons are produced with energies less than 0.1 MeV we can extend the limits on the integrals in Eq. (3.48) from zero to infinity without loss of generality. With this proviso, the normalization condition of Eq. (2.32) sets the denominator equal to one, and Eq. (3.48) reduces to

\[ \bar{\sigma}_{xF} = \int \sigma_x(E) \chi(E) dE. \] (3.49)

Table 3.2 lists fast cross sections averaged over the fission spectrum for several of the isotopes that appear most prominently in...
<table>
<thead>
<tr>
<th>Nuclide</th>
<th>Thermal Spectrum Cross Sections</th>
<th>Resonance Integrals</th>
<th>Fast (Fission Spectrum) Cross Sections</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\sigma_f$</td>
<td>$\sigma_a$</td>
<td>$\sigma_s$</td>
</tr>
<tr>
<td>$^1\text{H}$</td>
<td>0</td>
<td>0.295</td>
<td>47.7</td>
</tr>
<tr>
<td>$^2\text{H}$</td>
<td>0</td>
<td>$5.06 \times 10^{-4}$</td>
<td>5.37</td>
</tr>
<tr>
<td>$^{10}\text{B}$</td>
<td>0</td>
<td>3409</td>
<td>2.25</td>
</tr>
<tr>
<td>$^{11}\text{C}$</td>
<td>0</td>
<td>$3.00 \times 10^{-3}$</td>
<td>4.81</td>
</tr>
<tr>
<td>$^{16}\text{O}$</td>
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<td>$1.69 \times 10^{-4}$</td>
<td>4.01</td>
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<tr>
<td>$^{23}\text{Na}$</td>
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<td>0.472</td>
<td>3.09</td>
</tr>
<tr>
<td>$^{26}\text{Fe}$</td>
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<td>2.29</td>
<td>11.3</td>
</tr>
<tr>
<td>$^{40}\text{Zr}$</td>
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<td>0.16</td>
<td>6.45</td>
</tr>
<tr>
<td>$^{133}\text{Xe}$</td>
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<td>$2.64 \times 10^6$</td>
<td>—</td>
</tr>
<tr>
<td>$^{149}\text{Sm}$</td>
<td>0</td>
<td>$6.15 \times 10^4$</td>
<td>—</td>
</tr>
<tr>
<td>Nuclide</td>
<td>Thermal Spectrum Cross Sections</td>
<td>Resonance Integrals</td>
<td>Fast (Fission Spectrum) Cross Sections</td>
</tr>
<tr>
<td>---------</td>
<td>--------------------------------</td>
<td>---------------------</td>
<td>---------------------------------------</td>
</tr>
<tr>
<td></td>
<td>$\sigma_f$</td>
<td>$\sigma_a$</td>
<td>$\sigma_s$</td>
</tr>
<tr>
<td>$^{157}$Gd</td>
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<td>$1.92 \times 10^5$</td>
<td>1422</td>
</tr>
<tr>
<td>$^{232}$Th</td>
<td>0</td>
<td>6.54</td>
<td>11.8</td>
</tr>
<tr>
<td>$^{233}$U</td>
<td>464</td>
<td>506</td>
<td>14.2</td>
</tr>
<tr>
<td>$^{235}$U</td>
<td>505</td>
<td>591</td>
<td>15.0</td>
</tr>
<tr>
<td>$^{238}$U</td>
<td>$1.05 \times 10^{-5}$</td>
<td>2.42</td>
<td>9.37</td>
</tr>
<tr>
<td>$^{239}$Pu</td>
<td>698</td>
<td>973</td>
<td>8.62</td>
</tr>
<tr>
<td>$^{240}$Pu</td>
<td>$6.13 \times 10^{-2}$</td>
<td>263</td>
<td>1.39</td>
</tr>
<tr>
<td>$^{241}$Pu</td>
<td>946</td>
<td>1273</td>
<td>11.0</td>
</tr>
<tr>
<td>$^{242}$Pu</td>
<td>$1.30 \times 10^{-2}$</td>
<td>16.6</td>
<td>8.30</td>
</tr>
</tbody>
</table>

power reactor cores. Such cross sections, however, provide only a smoothed approximation to what the cross sections would be if averaged over the actual flux distribution.

**Resonance Cross Section Averages**

Often the terms intermediate and resonance are used interchangeably in describing the energy range between 1.0 eV and 0.1 MeV because as neutrons slow down from fast to thermal energy the large cross sections caused by the resonances in uranium, plutonium, and other heavy elements account for the nearly all of the neutron absorption in this energy range. Equation (3.29) provides a reasonable approximation to the flux distribution in this energy range. However, as in the fast spectrum, the \( \Sigma_t(E) \) term in the denominator is dependent on all of the constituents present in the reactor and thus must be eliminated in order to obtain cross sections that are independent of core composition. Ignoring the energy dependence of the total cross section, we simplify the flux to \( \varphi(E) \approx 1/E \). Equation (3.48) then becomes

\[
\bar{\sigma}_{xl} = \int_1^x \sigma_x(E) \frac{dE}{E} \left/ \int_1^x \frac{dE}{E} \right.
\]

(3.50)

For capture and fission reactions intermediate range cross sections are frequently expressed as

\[
\bar{\sigma}_{xl} = I_x \left/ \int_1^x \frac{dE}{E} \right.
\]

(3.51)

where

\[
I_x = \int \sigma_x(E) \frac{dE}{E}
\]

(3.52)

defines the resonance integral. Since the predominate contributions to \( I_x \) \( (x = a, f) \) arise from resonance peaks—such as those shown in Figs. 2.6, 2.9, and 2.10—that lie well within the range \( 1.0 \text{ eV} \leq E \leq 0.1 \text{ MeV} \), the values of resonance integrals are relatively insensitive to the limits of integration. The denominator of Eq. (3.51), however, depends strongly on those limits. Evaluating it between 1.0 eV and 0.1 MeV, then, yields

\[
\bar{\sigma}_{xl} = 0.0869 I_x.
\]

(3.53)

Table 3.2 includes the resonance integrals for common reactor constituents.

As the thermal reactor spectrum in Fig. 3.6 indicates, the \( 1/E \)—that is, the \( E \varphi(E) = \text{constant} \)—spectrum is a reasonable approximation
through the slowing down region. However, the dips that appear represent
the resonance self-shielding that decreases the number of neutrons that
are lost to absorption. Since Eq. (3.52) does not include the effects of self-
shielding, numbers listed in Table 3.2 only provide an upper bound on
resonance absorption, which would only be obtained in the limit of an
infinitely dilute mixture of the resonance absorber in a purely scattering
material. In reactor cores self-shielding dramatically reduces the
amount of absorption. Advanced methods for calculating resonance
absorption accurately are beyond the scope of this text. However,
Chapter 4 includes empirical formulas that provide reasonable approx-
imations to resonance absorption with the effects of self-shielding
included.

Thermal Cross Section Averages

Although accurate determination of the thermal spectrum also
requires advanced computational methods, averages over simplified
spectra often serve as a reasonable first approximation in performing
rudimentary reactor calculations. We approximate the thermal flux
with the Maxwell-Boltzmann distribution, \( \varphi(E) \approx \varphi_M(E) \), given by
Eq. (3.35). With the normalization proved by Eq. (3.36), Eq. (3.48)
thus reduces to

\[
\bar{\sigma}_{xt} = \int \sigma_x(E) \varphi_M(E) dE. \tag{3.54}
\]

Since \( \varphi_M(E) \) is vanishingly small for energies greater than an
electron volt, the upper limit on this integral can be increased
from 1.0 eV to infinity without affecting its value. Thermal neu-
tron cross sections averaged over the Maxwell-Boltzmann distribution at room temperature of 20 \(^\circ\)C [i.e., 293 K] are tabulated for
common reactor materials in Table 3.2. Appendix E provides a
more comprehensive table of microscopic thermal cross sections
integrated over the Maxwell-Boltzmann flux distribution as in Eq.
(3.54), along with molecular weights and densities. The table
includes all naturally occurring elements and some molecules
relevant to reactor physics.

Frequently the cross sections are measured at 0.0253 eV, which
corresponds to a neutron speed of 2200 m/s. The convention is based
on what follows. The maximum—or most probable—value of \( \varphi_M(E) \)
may easily be shown to be

\[
E = kT = 8.62 \times 10^{-5} T \text{ eV}, \tag{3.55}
\]
where \( T \) is in degrees kelvin. We take the corresponding neutron speed to be

\[
v = \sqrt{2E/m} = \sqrt{2kT/m} = 128\sqrt{T} \text{ m/s}.
\]  

(3.56)

Cross section measurements made at \( T_0 = 293.61 \) K yield \( E_o = 0.0253 \) eV and \( \nu_o = 2,200 \) m/s; 0.0253 eV and 2200 m/s are commonly referred to as the energy and speed of a thermal neutron. The cross sections tabulated in Table 3.2 and Appendix E, however, are the averages over the Maxwell-Boltzmann spectrum given by Eq. (3.54), rather than the cross sections evaluated at \( E_o = 0.0253 \) eV.

In the many cases where thermal scattering cross sections are independent of energy, Eq. (3.54) reduces to \( \bar{\sigma}_{sT} = \sigma_s \). At thermal energies, however, the binding of atoms to molecules or within crystal lattices can significantly affect the thermal scattering cross sections. To account for this, the cross sections for hydrogen, deuterium, and carbon given in Table 3.2 and Appendix E are corrected to include the effects of such binding. These corrections allow Eqs. (2.14) and (2.15), for example, to be used without modification in the determination of the thermal scattering cross section of water.

In contrast, many thermal absorption cross sections are proportional to \( 1/\nu \):

\[
\sigma_a(E) = \sqrt{E_o/E} \sigma_a(E_o).
\]  

(3.57)

To obtain the energy-averaged cross section in such cases we must substitute this equation and Eq. (3.35) into Eq. (3.54):

\[
\bar{\sigma}_{aT} = \int_0^\infty \sqrt{E_o/E} \sigma_a(E_o) \frac{1}{(kT)^2} E \exp(-E/kT) dE.
\]  

(3.58)

Evaluating the integral, we obtain

\[
\bar{\sigma}_{aT} = \sqrt{\frac{\pi}{2}} \left( \frac{E_o}{kT} \right)^{1/2} \sigma_a(E_o) = 0.8862(T_o/T)^{1/2} \sigma_a(E_o).
\]  

(3.59)

Thus the \( 1/\nu \) absorption cross section is dependent on the absolute temperature, and even if \( T = T_0 \) the averaged absorption cross sections are not the same as those measured at \( E_o \). In Table 3.2 and Appendix E, thermal absorption and capture cross sections are the averages defined by \( \bar{\sigma}_{aT} \) of Eq. (3.54).

To correct these \( 1/\nu \) thermal cross sections for temperature, we note from Eq. (3.59) that \( \bar{\sigma}_{aT}(T) = (T_o/T)^{1/2} \bar{\sigma}_{aT}(T_0) \). In dealing with macroscopic thermal cross sections correcting for temperature becomes more complex if the material has a significant coefficient
of thermal expansion. Since $\Sigma_x = N\sigma_x$, and the atom density is given
by $N = \rho N_0/A$, density decreases with increasing temperature will
also cause macroscopic cross sections to decrease even if the micro-
scopic cross sections remain constant.

### 3.6 Infinite Medium Multiplication

We conclude this chapter by returning to the calculation of the
multiplication, $k_\infty$, the ratio of the number of fission neutrons pro-
duced to the number of neutrons absorbed. The ratio is determined by
using the reaction rate definition, Eq. (3.13). Since the number of
fission neutrons produced is $\int_0^\infty \nu \Sigma_f(E) \varphi(E) dE$, where $\nu$
is the number of
neutrons/fission, and the number of neutrons absorbed is
$\int_0^\infty \Sigma_a(E) \varphi(E) dE$, we have

$$k_\infty = \int_0^\infty \nu \Sigma_f(E) \varphi(E) dE / \int_0^\infty \Sigma_a(E) \varphi(E) dE. \quad (3.60)$$

Using the definitions of the energy-averaged cross sections and flux
given in Eqs. (3.37) we may express $k_\infty$ as a ratio of cross sections:

$$k_\infty = \nu \Sigma_f / \Sigma_a, \quad (3.61)$$

where only fissionable materials contribute to the numerator, while
absorption cross sections of all of the reactor core’s constituents
contribute to the denominator.

Thus far we have assumed implicitly that the fuel, moderator,
coolant, and other core constituents are all exposed to the same
energy-dependent flux $\varphi(E)$. Provided the volumes of core constituents
are finely mixed—for example, powders of uranium and graphite—this
assumption holds. However, in power reactors the diameters of fuel
elements, the spacing of coolant channels, and geometric configurations
of other constituents result in larger separations between materials. In
these circumstances the flux magnitudes to which the fuel, coolant,
and/or moderator are exposed often are not identical. Power reactor
cores consist of lattices of cells, each consisting of a fuel element, cool-
ant channel, and in some cases a separate moderator region. The expres-
sions derived above remain valid provided we interpret them as spatial
averages over the constituents of one such cell, with account taken for
differences in flux magnitudes. In the following chapter we first examine
the lattice structures of power reactors. We then take up the modeling of
fast and thermal reactor lattices in order to examine these differences in
flux magnitudes and then to obtain expressions for $k_\infty$ explicitly in terms of the various core constituents.

**Bibliography**


**Problems**

3.1. Verify Eqs. (3.23) and (3.25).

3.2. Show that in Eq. (3.31) the normalization condition $\int^E_{E_0} s(E)\,dE = 1$ must be obeyed. Hint: Note that $\int^E_{E_0} p(E' \to E)\,dE = 1$ for $E' \leq E_0$.

3.3. In Eq. (3.31) suppose that the neutron slowing down past $E_0$ is due entirely to elastic scattering from a single nuclide with $A > 1$, and with no absorption for $E > E_0$. Show that $s(E)$ then takes the form

$$s(E) = \begin{cases} \frac{1}{(1-\alpha)\xi} \left( \frac{1}{E_0} - \frac{\alpha}{E} \right), & \alpha E_0 < E < E_0, \\ 0, & E < \alpha E_0 \end{cases}$$

3.4. For thermal neutrons calculate $\bar{\eta}$ as a function of uranium enrichment and plot your results. Use the uranium data from the following table:

<table>
<thead>
<tr>
<th></th>
<th>$\nu$</th>
<th>$\sigma_f$ (barns)</th>
<th>$\sigma_a$ (barns)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Uranium-235</td>
<td>2.43</td>
<td>505</td>
<td>591</td>
</tr>
<tr>
<td>Plutonium-239</td>
<td>2.90</td>
<td>698</td>
<td>973</td>
</tr>
<tr>
<td>Uranium-238</td>
<td>—</td>
<td>0</td>
<td>2.42</td>
</tr>
</tbody>
</table>
3.5. Suppose a new isotope is discovered with a “1/E” absorption cross section given by $\Sigma_a(E) = (E_0/E)\Sigma_a(E_0)$. Determine the energy-averaged cross section if the isotope is placed in the thermal flux distribution given by Eq. (3.35).

3.6. In the wide resonance approximation (also called narrow resonance infinite mass approximation because the fuel is assumed to have an infinite mass), $A' \rightarrow \infty$ and thus $\alpha' \rightarrow 1$ in the first integral on the right of Eq. (3.28) while the remaining approximations are the same as in narrow resonance approximation. Determine $\varphi(E)$ through the resonance. How does it differ from Eq. (3.29)? In which case is there more energy self-shielding?

3.7. Lethargy defined as $u = \ln(E_0/E)$ is often used in neutron slowing down problems; lethargy increases as energy decreases. Note the following transformations: $\varphi(E)dE = -\varphi(u)du$, $p(E \rightarrow E')dE' = -p(u \rightarrow u')du'$, and $\Sigma_x(E) = \Sigma_x(u)$.

a. Show that $p(E \rightarrow E')$ given by Eq. (2.47) becomes

$$p(u \rightarrow u') = \begin{cases} \frac{1}{1-\alpha}\exp(u-u'), & u \leq u' \leq u + \ln(1/\alpha), \\ 0, & \text{otherwise} \end{cases}$$

b. Express Eq. (3.22) in terms of $u$.

3.8. Making a change of variables from energy to speed, show that Eq. (2.47) becomes

$$p(v \rightarrow v') = \begin{cases} \frac{2v'}{(1-\alpha)v^2}, & v\sqrt{\alpha} \leq v' \leq v, \\ 0, & \text{otherwise} \end{cases}$$

3.9. Suppose that the Maxwell-Boltzmann distribution, Eq. (2.34), represents the neutron density in Eqs. (3.43) and (3.44):

a. Find the value of $\bar{v}$.

b. If we define $\bar{E} \equiv \frac{1}{2}mv^2$, show that $\bar{E} = 1.273kT$.

c. Why is your result different from the average energy of $\frac{3}{2}kT$ given by Eq. (2.33)?
3.10. A power reactor is cooled by heavy water \([D_2O]\) but a leak causes a 1.0 atom \(\%\) contamination of the coolant with light water \([H_2O]\). Determine the resulting percentage increase or decrease in the following characteristics of the coolant:

a. Slowing down decrement.
b. Slowing down power.
c. Slowing down ratio.

3.11. Using the data in Appendix E calculate the microscopic absorption cross section of water, averaged over a thermal neutron spectrum:

a. At room temperature.
b. At 300\(^\circ\)C, which is a typical operating temperature for a water-cooled reactor.

3.12. Repeat problem 3.11 for heavy water.