QCD plasma instability and thermalization

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Stages of heavy ion collision:

- **1 TeV/A, \( \gamma \sim 1000 \)**

- **\( \tau < 0 \): initial state:**
  Color Glass Condensate (CGC),
  with characteristic momentum scale:
  saturation scale \( Q_s \gtrsim \) few GeV

- **\( \tau \sim 0.1 \text{ fm} \): “melting” of CGC; excitations with \( p \sim Q_s \)
  – anisotropic & non-thermal initial distribution!

- **\( \tau \lesssim 1 \text{ fm} \): Very rapid isotropization & thermalization
  (observed at RHIC) (topic of this talk!)**

- **\( 1 \lesssim \tau \lesssim 10 \text{ fm} \): Expansion of \( \sim \) thermal quark-gluon plasma (QGP)**

- **\( \tau \sim 10 \text{ fm} \): hadronisation**
Heavy-Ion collisions & hard modes

- $\tau \approx 1/Q_s$: In initial stages of HIC the “plasma” consists of hard ($p_{\text{hard}} \sim Q_s$) modes.
- $\tau \gg 1/Q_s$: As the system expands, the hard mode distribution becomes dilute (perturbative),

$$n_{\text{hard}} \ll p_{\text{hard}}^3/g^2,$$

and it becomes squeezed along $z$-direction (free streaming).

[Baier, Mueller, Schiff, Son]

- Dilute & $Q_s \gg \Lambda_{\text{QCD}} \rightarrow$ hard modes behave like on-shell classical particles.
Rapid thermalization

What turns the very non-thermal hard mode distribution to $\sim$ thermal (isotropic) so quickly?

- **Bottom-up thermalization: hard-hard collisions** [Baier, Mueller, Schiff, Son, ...]
  - Achieve isotropization in $\tau \sim \alpha_s^{-13/5}/Q_s$: 2–4 fm?

- **Plasma instabilities:**
  - Well-known in electrodynamics (non-trivial current distributions)
  - Can happen in QCD too: non-isotropic hard mode distribution
    - exponential growth of soft modes ($p \ll Q_s$), plasma instability
    - strong back reaction to hard modes
    - thermalization
  - [Mrówczyński; Mrówczyński, Strickland; Arnold, Lenaghan, Moore; Romatschke, Strickland; ...]
  - Parametrically (in $g$) faster than collisions above
Weibel instability

- In electromagnetic plasmas, anisotropic distribution of current carrier distribution (electrons) which leads to **Weibel (filamentary) instability**:

  \[ B \]

  \[ I \]

  \[ \Rightarrow \] Exponential growth of soft magnetic fields; \( p_{\text{soft}} \ll p_{\text{electron}} \). In QED the growth rate can be solved analytically as a function of the anisotropy.

  \[ \Rightarrow \] When magnetic field amplitude is large, \( gA_{\text{soft}} \sim k_{\text{electron}} \), field bends electrons strongly \( \Rightarrow \) isotropization, thermalization?

- Should play a role in heavy ion collisions too? [Mrówczyński; Arnold, Lenaghan, Moore; Strickland]
Weibel instability in HICs

- QED $\rightarrow$ QCD:
  - electrons $\rightarrow$ hard gluons
  - soft electromagnetic field $\rightarrow$ soft gluons

- Small-amplitude soft fields ($f_{\text{soft}} \ll g^2$): the growth rate can be solved analytically; essentially QED (non-abelian commutators can be neglected)

$\Rightarrow$ exponential growth of soft fields, with characteristic $k_{\text{soft}} \sim k^*$

- What happens when magnitude of the soft fields reach the “non-abelian limit” $gA_{\text{soft}} \sim k^*$ (or $f_{\text{soft}} \sim g^2$)?
  - Continued growth until $gA_{\text{soft}} \sim p_{\text{hard}}$ (as in QED), leading to efficient isotropization?
  - Just stops? Not so efficient
  - Something else?

- Continued growth may be possible if the fields ‘Abelianise’, i.e. only one colour component grows. [Arnold, Lenaghan, Moore]

- Special lattice simulations needed.
How to study the system?

- **Soft fields:** non-perturbative, large occupation numbers \((f_{\text{soft}} \gg g^2)\):  
  \(\sim\) classical evolution

- **Hard modes:** dilute, weakly coupled  \(\sim\) classical particles  

\[ \Rightarrow \]

**A)** Classical pure gauge field evolution  
[Romatschke,Venugopalan; Berges,Scheffler,Sexty]

**B)** System with hard "classical" particles + soft non-perturbative gauge fields ("HTL" theory)

**B1)** Real particles  
[Dumitru,Nara,Strickland]

**B2)** Particle distribution functions, "\(W\)"-fields  
[Arnold,Moore,Yaffe; Rebhan,Romatschke,Strickland; Bödeker,KR]  
Fixed anisotropic background distribution + fluctuations \((W)\)
“Classical gauge”:
- All scales need to fit: large lattices
- No overcounting
- Feedback hard ↔ soft, full isotropization possible
- Total energy conserved

“Particles”:
- Separation of scales
- Feedback hard ↔ soft
- Total energy
- overcounting?

“$W$-fields”:
- Static anisotropic background + dynamic fluctuations
  ⇒ Full isotropization not possible
- Separation of scales
- Technically “clean”
Hard Thermal Loop effective theory

Hard modes behave as on-shell particles moving in soft background fields, with a distribution function

\[ f_{\text{hard}}(x, \vec{p}) = \tilde{f}(\vec{p}) + \lambda^a f^a(x, \vec{p}) + \ldots \]

where the singlet \( \tilde{f}(\vec{p}) \) is constant in space and time, and is anisotropic.

Yang-Mills-Vlasov equations of motion:

\[
(D_\mu F^{\mu\nu})^a = J^{a,\nu}_{\text{hard}} = g \int \frac{dp}{p^3} v^\nu f^a
\]

\[
(v \cdot Df)^a + g v^\mu F^{a}_{\mu i} \frac{\partial \tilde{f}}{\partial p^i} = 0
\]

where \( v = (1, \vec{p}/p) \). Defining \( W \)-function

\[
W^a(x, \vec{v}) \equiv 4\pi g \int_0^\infty \frac{dp p^2}{(2\pi)^3} f^a(x, \vec{p})
\]

we can integrate EQM over \(|p|\), obtaining \ldots
Hard Thermal Loop effective theory

Yang-Mills-Vlasov EQM:

\[
(D_\mu F^{\mu\nu})^a = \int \frac{d\Omega}{4\pi} v^\nu W^a \\
(v \cdot DW)^a = m_0^2 v^\mu F^{ai}_\mu U^i(\vec{v})
\]

where \( U^i(\vec{v}) \) characterises the anisotropic \( \bar{f} \):

\[
m_0^2 U^i(\vec{v}) = -4\pi g^2 \int_0^\infty \frac{dpp^2}{(2\pi)^3} \frac{\partial \bar{f}(p\vec{v})}{\partial p^i}
\]

For isotropic \( \bar{f} \) we have \( \bar{U} = \vec{v} \), and \( m_0 = m_{\text{Debye}} \). \( m_0 \) is the only dimensionful parameter.
Lattice simulations

- The hard mode distribution is modelled with $W^a(x, \vec{v})$ fields. These are expensive: live on $R^3 \times S^2$.
- $\vec{v}$ dependence modelled in 2 ways:
  - expansion in spherical harmonics
    [Bödeker, Moore, K.R.; Arnold, Moore, Yaffe; Bödeker, K.R.]
  - sample discrete directions
    [Rebhan, Romatschke, Strickland]
- We use spherical harmonic “$W$-fields”, SU(2) gauge group
- We use similar techniques than Arnold, Moore, Yaffe, but with
  - 5 different values for the anisotropy, both weaker and much stronger than AMY
  - Large lattices (up to $240^3$), with a large number of auxiliary $W$-fields (up to $L_{\text{max}} = 48$, i.e. 14250 auxiliary fields in addition to $A_{\mu}^a$).
On the lattice:

- We expand $W$, $\bar{f}$ in spherical harmonics:

$$W^a(x, \vec{v}) = W_{\ell m}^a Y_{\ell m}(\vec{v}),$$
$$\bar{f}(\vec{p}) = \bar{f}_{\ell m}^a(p) Y_{\ell m}(\vec{v}),$$

where $\ell = 0 \ldots L_{\text{max}}$.

- We use $A_0 = 0$ gauge

- The dynamical lattice fields are $U_i \in \text{SU}(2)$, $E_i^a$, $W_{\ell m}^a$

- $m_0$ dimensionful; lattice spacing given by $am_0$.

- 4 lattice “cutoff” artifacts:
  - finite lattice spacing $a \to 0$
  - finite volume $L^3 \to \infty^3$
  - finite $L_{\text{max}} \to \infty$
  - finite timestep $\delta t \to 0$
Anisotropic hard mode distributions

We parametrise the anisotropic hard mode distributions by expanding in spherical harmonics:

\[ \bar{f} = \sum_{\ell=0}^{L_{\text{asym}}} f_{\ell0} Y_{\ell0}, \]

with \( L_{\text{asym}} = 2 \ldots 28 \). For each \( L_{\text{asym}} \) we try to maximally localise the distribution along \( xy \)-plane:

- “Propellor”-shaped distributions
- Asymmetry parameter

\[ \xi^2 = \frac{\langle v_z^2 \rangle}{\langle v_{\perp}^2 \rangle} = 0.5 \ldots 0.015 \]

Naturally \( L_{\text{asym}} < L_{\text{max}} \).
Growth rate in U(1) (weak field)

- Growth rate as a function of $k$
- Much wider range of diverging wave vectors at large asymmetry (large $L_{\text{max}}$)

$k/m_0$ vs. $B^2$ growth rate / $m_0$
Growth rate in U(1) (weak field)

- Growth rate as a function of $k$
- Much wider range of diverging wave vectors at large asymmetry ($\text{large } L_{\text{max}}$)
- Max growth rate varies from $\sim 0.15 \ldots 0.8/m_0$
- Location of maximal growth $k^* \sim m_0$. 

![Diagram showing growth rate as a function of $k/m_0$ with labels indicating peaks and maxima at $k = k^*$ max growth rate.](image)
$L_{\text{max}}$ dependence (U(1) or weak field)

$L_{\text{asym}} = 2$

$L_{\text{asym}} = 28$

$L_{\text{max}}$ cutoff effects small in practice!
What we observe:

Small initial fields, exponential (analytically solvable) growth at a wave vector \( k^* \ll p_{\text{hard}} \). What happens when \( gA_{k^*} \sim k^* \), or \( B^2 \sim k^{*4}/g^2 \)?
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Abelianisation and continued exponential growth at $k \sim k^*$ [Arnold, Lenaghan, Moore]

- Not seen in QCD; QED OK
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1. Abelianisation and continued exponential growth at $k \sim k^*$ [Arnold, Lenaghan, Moore]
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2. Exponential growth stops, diffusion to UV with slow linear growth (no thermalization)
   - Weak to moderate anisotropy [Arnold, Moore, Yaffe]
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3. Growth of \( A_{k^*} \) stops, rapid avalanche to UV with \( \sim \) exponential growth of energy
   - We observe this at strong anisotropy
   - almost full saturation of lattice modes \( \Rightarrow \) direct thermalization?
Generic growth of energy:

\[ L_{\text{max}} = 32, \ L_{\text{asym}} = 28, \ m_0 a = 0.3, \ 128^3 \]

For our values of asymmetry, system becomes non-abelian when \( \frac{1}{2} B^2 \sim \frac{(k^*)^4}{4g^2} \).
For our values of asymmetry, system becomes non-abelian when $\frac{1}{2}B^2 \sim \left(\frac{(k^*)^4}{4g^2}\right)$.
Results: growth of energy with small anisotropy

- Little growth seen beyond the weak field region at $L_{\text{max}} = 2, 4$
- lattice UV modes far from saturated
- very small lattice spacing dependence
- agrees with Arnold, Moore, Yaffe ($L_{\text{max}} = 6$)
Results: growth of energy with large anisotropy

- Continued exponential growth in strong field region at $L_{\text{max}} = 14, 28$
- Stops when lattice UV modes saturate: $a$ dependence
- How far does it continue when $a \to 0$?
Results: growth of the saturation scale

Magnetic field energy density \( \left( \frac{1}{2} B^2 \right) \) when the exponential growth stops:

- Both for \( L_{\text{asym}} = 14, 28 \) the scale grows with a power of lattice spacing \( a \)
  - Growth regulated by \( a \)
  - Exponential avalanche to far UV in the continuum limit
  - Thermalization?
Results: gauge field spectrum

Fixing to Coulomb gauge, \( f_k \propto k \langle A_k^2 \rangle \)
Results: gauge field spectrum

Fixing to Coulomb gauge, $f_k \propto k\langle A_k^2 \rangle$

![Graph showing energy density and $f(k)$ against $k/m_0$.]

Fri May 5 10:28:04 2006
Results: gauge field spectrum

Fixing to Coulomb gauge, $f_k \propto k \langle A_k^2 \rangle$
Results: gauge field spectrum

Fixing to Coulomb gauge, \( f_k \propto k \langle A^2_k \rangle \)
Results: gauge field spectrum

Fixing to Coulomb gauge, \( f_k \propto k \langle A_k^2 \rangle \)

- Energy density / \( m_0^4 \)
- \( B^2/2 \)
- \( E^2/2 \)

- \( f(k) \)
- \( k^*/k_0 \)
- max. growth rate

Fri May 5 10:30:12 2006
Results: gauge field spectrum

Fixing to Coulomb gauge, $f_k \propto k \langle A_k^2 \rangle$
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\[ B^2/2 \]
\[ E^2/2 \]

\[ \text{Energy density / m}^4 \]

\[ \text{t m}_0 \]

\[ \text{max. growth rate} \]

\[ \text{k/m}_0 \]

Fri May 5 10:32:18 2006
Results: gauge field spectrum

Fixing to Coulomb gauge, \( f_k \propto k \langle A_k^2 \rangle \)

\[ \begin{align*}
B^2/2 & \quad 10^2 \\
E^2/2 & \quad 10^1 \\
\end{align*} \]

\[ \begin{align*}
\text{Energy density / m}_0^4 & \quad 10^0 \\
\end{align*} \]

\[ \begin{align*}
t \text{ m}_0 & \quad 0 \quad 20 \quad 40 \quad 60 \\
\end{align*} \]

\[ \begin{align*}
f(k) & \quad \text{non-abelian point} \\
\end{align*} \]

\[ \begin{align*}
\text{max. growth rate} & \quad k^*/2 \\
\end{align*} \]

\[ \begin{align*}
k/m_0 & \quad 0 \quad 2 \quad 4 \quad 6 \quad 8 \quad 10 \\
\end{align*} \]
Results: gauge field spectrum

Fixing to Coulomb gauge, \( f_k \propto k \langle A_k^2 \rangle \)
Results: gauge field spectrum

Fixing to Coulomb gauge, \( f_k \propto k \langle A_k^2 \rangle \)

![Graph showing the energy density and energy spectrum](image)

Fri May 5 10:37:02 2006
Fixing to Coulomb gauge, \( f_k \propto k \langle A_k^2 \rangle \)
Results: gauge field spectrum

Fixing to Coulomb gauge, $f_k \propto k \langle A_k^2 \rangle$
Results: gauge field spectrum

Fixing to Coulomb gauge, $f_k \propto k \langle A_k^2 \rangle$

The final spectrum is $\sim$ thermal ($f_k \propto 1/k$)
Small anisotropy remains IR dominated

- Exponential growth stops without full UV saturation.
- Slow $\sim$ linear growth

![Graph showing energy density vs. $m_0 t$ and $k/m_0$](image)
Growth of individual modes

**Strong anisotropy**
\[ L_{\text{asym}} = 28: \text{growth of modes } n k^*, n = 1 \ldots 10 \]

**Weak anisotropy**
\[ L_{\text{asym}} = 4: \text{growth of modes } n k^*, n = 1 \ldots 11 \]

(See also [Berges, Scheffler, Sexty])
Why UV modes grow so rapidly?

*Shape of the spectrum:*

- Spectrum looks like \( A_k \sim e^{-\alpha k} \) in the “Strong field” domain. At \( k \gg k^* \), growth caused by non-linear (commutator) terms in EQM
  \[ \Rightarrow \partial_i A \sim \partial_0 A \sim gA^2 \]
  \[ \Rightarrow kA_k \sim \partial_0 A_k \sim g \int_{k'} A_{k'} A_{k-k'} \approx g(A_{k/2})^2 \]
  \[ \Rightarrow A_k \sim e^{-\alpha k(t_f-t)}, \]
  where \( t < t_f \) and \( \alpha = O(1) \).

- Exponential shape, growth rate \( \propto k \). ~ OK.

*What powers the non-linear exponential growth?*

- Exponential flow of energy from hard modes to soft fields \( \Rightarrow \) some kind of instability must still be active.

- Not like the linear (Weibel) instability! Different characteristics, mechanism unknown.

- Gauge fixing artifacts? Checked with gauge-invariant measurements (e.g. cooling).
Results: isotropization

\[ L_{\text{max}} = 32, \quad L_{\text{asym}} = 28, \quad m_0 a = 0.3, \quad 128^3 \]

Soft fields become nearly isotropic when entering the “Strong field” domain; fully isotropic after UV saturation.
$B^a$ (1 color component) along $\perp$-plane
$B^a$ (1 color component) along $\perp$-plane
$B^a$ (1 color component) along $\perp$-plane
$B^a$ (1 color component) along $\bot$-plane
$B^a$ (1 color component) along $\perp$-plane
$B^a$ (1 color component) along $\perp$-plane
Large initial fields

- The growth is suppressed if the initial amplitude of soft fields is too large!
- Initial condition: random $E_i(k)$ with amplitude
  $$E_i(k) \sim C e^{-k^2/(2m_0)^2}$$
- Vary $C$ →
- Linear growth with very weak initial fields generate favourable conditions for further (non-linear) growth!
- Energy slowly “cascades” to UV [Arnold, Moore]
- Needs further study
Lattice artifacts are under control:

- **Finite $a$ effects:**
  - small at small anisotropy
  - large at large anisotropy (UV avalanche)
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- **Finite volume effects:**
  - $L \gtrsim 5 \lambda^*$, where $\lambda^* = 2\pi/k^*$

---

\[ L_{\text{max}} = 16 \quad L_{\text{asym}} = 14 \quad m_0^2 a = 0.55 \]

\[ L = 3.9 \lambda^* \]
\[ L = 5.2 \lambda^* \]
\[ L = 7.8 \lambda^* \]
\[ L = 10.3 \lambda^* \]

\[ B^2/a^4 \]

\[ m_0^2 t \]

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- Finite $L_{\text{max}}$ effects:
  - in control when $L_{\text{max}}$ large enough
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- **Finite \( L_{\text{max}} \) effects:**
  - in control when \( L_{\text{max}} \) large enough

- **Finite timestep effects:**
  - negligible with \( \delta t = 0.05a \) and \( 0.1a \)
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- **Finite volume effects:**
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- **Statistics of one:**
  - only 1 or 2 runs for each parameter set
  - OK, because statistical variation $\ll$ physical variation
Conclusions

- We observe a fast growth in UV part of the soft fields if the asymmetry of the hard mode distribution is large enough.
- Growth fastest to $\hat{z}$-direction: “soft” modes fill up the $\hat{z}$ deficit in hard modes?
- Rate itself is sufficient for rapid thermalization. For large anisotropy

$$\text{rate} \sim m_0 \rightarrow m_{\text{Debye}} \Rightarrow \text{growth rate less than } 1/fm.$$  

- Warrants further study!
- Open problems:
  - Right initial field configuration?
  - Expanding system tends to slow down the onset of growth further
    [Romatschke,Venugopalan; Strickland, Nara, Rebhan]
UV runoff in compact U(1)

- compact lattice U(1) becomes non-linear when we hit the lattice limit $A_k \sim a^{-4} k^{-2}$. Causes runoff to UV too!
- Check signature by directly simulating compact U(1):
- Fourier spectrum: $f_{k,\text{max}} \gg 1$
UV runoff in compact $U(1)$

- compact lattice $U(1)$ becomes non-linear when we hit the lattice limit $A_k \sim a^{-4} k^{-2}$. Causes runoff to UV too!
- Check signature by directly simulating compact $U(1)$:
  - Fourier spectrum: $f_{k,\text{max}} \gg 1$
  - $f_{k,\text{max}}$ diverges when $a \to 0$. Very different behaviour wrt. non-Abelian theory!
Results: where is the energy?

\[ L_{\text{max}} = 32, \ L_{\text{asym}} = 28, \ m_0 a = 0.3 , \ 128^3 \]

- Initial equipartition due to white noise initial state; i.e. each lattice mode equally populated
- Weak field growth: energy in modes with \( k \sim k^* \)
- Strong field growth: energy runs to UV
- Approaches lattice equipartition
Results: where is the energy?

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- Weak field growth: energy in modes with \( k \sim k^* \)
- Strong field growth: energy runs to UV
- Approaches lattice equipartition
- UV divergent, depends on lattice spacing
Results: checking the gauge fixing

\[ L_{\text{max}} = 32, \; L_{\text{asym}} = 28, \; m_0 a = 0.3, \; 128^3 \]

- Gauge fixing always suspect with large fields and/or IR modes due to Gribov copies.
- Compare gauge fixed
  \[ \langle k^2 \rangle = \int dk \; k^2 f_k \]
  with gauge invariant
  \[ \langle k^2 \rangle = \frac{\langle [D_i F_{ij}]^2 \rangle}{\langle F_{ij}^2 \rangle} \]
- works well!